REPORT RESUMES

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PRELIMINARY DEVELOPMENT OF A JUNIOR-HIGH COURSE IN THE LOGICAL FOUNDATIONS OF PHYSICAL SCIENCE. (TITLE SUPPLIED)
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AN INTRODUCTORY, SEVENTH-GRADE PHYSICAL SCIENCE COURSE WAS CONSTRUCTED IN AN ATTEMPT TO DEVELOP A LOGICAL AND COHERENT SEQUENCE OF THE MATERIAL, DEVELOP AN APPRECIATION FOR PRECISENESS OF LANGUAGE AND LOGICAL THINKING, SHOW THE ROLE OF MATHEMATICS IN PHYSICAL SCIENCE AS AN APPLICATION OF LOGIC TO PHYSICAL EXPERIENCE, REVEAL THE SCIENTIST'S SPIRT OF INQUIRY TO THE STUDENT, TEACH THE MEANING AND PHILOSOPHY OF MEASUREMENT, AND DISCLOSE THE NATURE OF PHYSICAL SCIENCE AS A SEARCH FOR FUNCTIONAL RELATIONSHIPS AMONG PHYSICAL OBSERVABLES. A TEXTBOOK AND A LABORATORY MANUAL WERE DEVELOPED WHICH COVER THE FIVE SUBSTANTIVE TOPICS OF (1) NATURE OF MEASUREMENT, (2) CONSTANTS, VARIABLES, AND EQUATIONS, (3) FUNCTIONS AND PROPORTIONALITY, (4) WEIGHT, VOLUME, AND DENSITY, AND (5) MOTION IN A STRAIGHT LINE. SUGGESTIONS FOR FUTURE RESEARCH INCLUDED THE PREPARATION OF SIX ADDITIONAL UNITS OF SUBSTANTIVE MATERIAL, AND A TEST AND EVALUATION OF THE PRESENT MATERIALS. (GD)

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What is Physical Science?

Physical science is the study of matter and its behavior. Since the behavior of matter often involves forces, energy, temperature, speed, and many other things, all these belong also to the study of physical science. There are, of course, many different kinds of matter -- like iron, glass, water, air, sand, and gasoline. Physical science is concerned with kinds of matter, too. Sometimes one kind of matter may be changed into another kind. For example, wood may be changed into ashes by burning it, or a raw egg may be changed into something quite different by cooking it. Changes that matter undergoes are also part of the study of physical science.

If we push a brick off the top of a tall building, it falls to the ground. This is something that happens to a piece of matter, and the study of falling bodies is therefore of course part of physical science. But notice that just as a brick will fall from the top of the building, so will a horseshoe, a cuckoo clock, a parasol, or a wad of newspaper. In other words, the business of falling under gravity is common to all kinds of matter. Similarly, if you put a brass door-knob, a stick of wood, a diamond ring, or a golf-ball in a lighted oven, they all get hot. The business of getting hot in a warm oven is also common to all kinds of matter.

On the other hand, if you try to burn a sheet of paper and a sheet of iron, you find that only the paper will burn. If you drop a sheet of iron and a sheet of gold in a glass of acid, the iron will dissolve but not the gold. A pill of aspirin will relieve your headache but a pill of sugar will not. Vinegar will curdle milk but water will not. A raw egg will change greatly when dropped in boiling water but a golf ball will not. In other words we recognize that some sorts of physical happenings depend on the kind of matter you are talking about. Try to list a few physical happenings that apply to any kind of matter and a few others that apply only to certain kinds of matter.

It is customary to divide physical science into two main divisions. Those physical happenings where the kind of matter involved is not important to the discussion are usually said to belong to the study called physics. Those physical happenings where the kind of matter is important, or where the kind of matter you start with changes to another kind, belong to the study called chemistry. But the distinction between physics and chemistry is a very fuzzy one and not at all important. No scientist could possibly tell you the exact difference between physics and chemistry because there simply is no fence between the two.

Some people have the mistaken idea that physical science is basically a hard subject. This is totally untrue. The basic ideas of physical science are very simple. You will have no trouble with them at all. The only difficulty that people ever have with physical science is really a difficulty with English. If you say "I only have two pencils" when you mean "I have only two pencils", you may have trouble. In the same way, you should clearly understand how each of the following pairs of sentences, sometimes used as though they mean the same thing, really differ in meaning:

I don't like spinach.	I dislike spinach.
I gave the wrong answer to every question on the test.	I didn't give the right answer to any question on the test.
Every shark is not a man-eater.	Not every shark is a man-eater.
Every gink is a foople.	Every foople is a gink.
Did anyone forget to bring their lunch today?	Did anyone forget to bring his lunch today?
I painted all of the boats.	I painted all the boats.
The recipe calls for five teaspoons full of sugar.	The recipe calls for five teaspoonfuls of sugar.
I was so hungry for cake that I ate the whole recipe.	I was so hungry for cake that I ate the whole cake.
My family like to go on picnics.	My family likes to go on picnics.
Your sister is a beautiful dancer.	Your sister dances beautifully.
I don't have no money in my pocket.	I don't have any money in my pocket
Write nothing on the blackboard.	Write "nothing" on the blackboard.
I like the boys playing in the yard.	I like the boys' playing in the yard.

Notice that none of these sentences is incorrect. It is simply that the two sentences in each pair have different meanings. Try to explain how the meanings differ.

If you can see clearly the differences between the meanings of the sentences above and can learn to use English correctly, then you will have no trouble with physical science. The one absolute necessity in learning of physical science is the correct use of English -- in reading, in writing, and in speaking. All the rest is easy.

But you ought to be warned at the beginning of one important thing. You cannot expect to read and understand a science book as fast as you can a story book or a comic book. Read only as fast as you understand what you are reading. Don't be ashamed to go back and read a difficult sentence as many times as you have to to understand it. If you skip or fail to understand a sentence -- or even a whole paragraph! -- in a story, you usually can pick up the story without loss. But you cannot often do that in this book!

Unit I.

Making Measurements

1. Comparisons

Everyone has heard of comparisons and everyone makes comparisons every day. When you say "John is taller than Mike" - you are making a comparison between John's height and Mike's height and stating that John's height is the greater. Here are somecomparisons the like of which you have probably yourself made at one time or another:

I have more marbles than Sam.

Mr. Smith's car is faster than Mr. Brown's.

Charlie is heavier than Sue.

Molly lives farther from school than Chuck.

A milk bottle holds more than a salt-shaker.

Your living-room floor has more area than a sheet of notebook paper.

A tractor can pull harder than a rabbit.

It was warmer yesterday than it is today.

A right angle is larger than the angle at the point of a sharpened pencil.

Try your hand at writing out some comparisons like these. Try to make them comparisons of different kinds of things.

In each of the above examples, notice that the sentence first calls your attentio to some kind of quality that is possessed by two people or things. The sentence then reminds you that the two people or things possess this quality to different degrees. A finally it tells you which of them possesses it to the greater degree.

For instance, the first example comparing the number of Sam's marbles with mine says something like this: "Everybody has some number of marbles (Remember that zero is a number!). It is possible to find out this number both for me and for Sam. If you find out both Sam's and my number, you will see that mine is the larger." In this case, the "quality" we are talking about and comparing is "number of marbles." For the examples above, these are the qualities being compared:

Number	(of marbles)
Speed	(of cars)
Weight	(of persons)
Distance	(of a point from other points)
Volume	(of containers)
Area	(of surfaces)
Force	(of things used to pull things)
Temperature	(of the air on different days)
Angle	(between pairs of lines)

Be sure you understand that the sentence-comparisons above speak of just the qualities in this list, and then list the qualities dealt with in the comparisons that you wrote yourself.

Now examine these comparison-sentences:

I have more influence than Sam.

Mr. Smith's car is nicer-looking than Mr. Brown's.

Charlier is healthier than Sue.

Molly's house is more pleasant than Chuck's.

A milk bottle is better than a salt shaker.

We had more fun yesterday than today.

At first sight, these comparisons look much like the first group, but there is a very important difference. You can best see this difference by looking at the list of qualities in the sentences.

Influence	(of persons)
Niceness of appearance	(of cars)
Health	(of persons)
Pleasantness	(of houses)
Goodness	(of containers)
Fun	(of a person on different days)

Do you see what the qualities in the first list have that the qualities in the second do

If you do not clearly see the difference, think of it this way. Think about the sentence about marbles above as an example. In order to find out whether it is true that I have more marbles than Sam, all I have to do is to compare the number of marbles I have with the number of marbles Sam has. O.K., how many marbles do I have? By actual count, I have 87. Sam has 74. You know that 87 is greater than 74 and the sentence comparison is therefore correct.

Or, when we talk about the speeds of cars, we can by actual trial find out how fast Mr. Smith's and Mr. Brown's cars can go. If Mr. Smith's car can go 85 miles per hour (not on a public highway, of course) and Mr. Brown's only 78, the case is proved. Also, you can find out how many pounds Charlie and Sue each weighs, how many miles Molly and Chuck live from school, and how many teaspoonfuls of water the milkbottle and the salt-shaker each holds. How many square feet to your living-room floor? How many pounds can the tractor pull? What was the temperature yesterday? How many degrees in the angle of a pencil point?

Notice that all these questions can be answered. But can you really give an answer to such questions as "How much influence does Sam have?", "How nice does Mr. Brown's car look?", "How good is a milkbottle?", and "How much fun did we have yesterday?"? These questions have some meaning, of course, but neither the questions nor the possible answers to them have the precision of which the others are capable.

You recognize then that some qualities are very special in that they can be measured or counted. Length, number, volume, weight, etc. -- all those in the first list above, and many more besides -- are such qualities. Tell how you might go about measuring or counting each quality in the list. On the other hand, there are other qualities -- like influence, niceness of appearance, pleasantness, and many more -- that cannot be measured or counted.

When a quality is measurable or countable, its measure (or count) is called a quantity. For instance, a count of 87 (marbles) is a quantity. A speed of 56 miles per hour is a quantity. So are a weight of 105 pounds, a distance of 1 1/2 miles, a volume of 1 quart, an area of 272 square feet, a force of 31.2 pounds, a temperature of 72°F, and an angle of 15°. Each of the quantities we have met so far consists either of a number or a number plus a unit. The quantity of 87 marbles is expressed by the number 87 alone. The quantity expressing Charlie's weight, however, must be expressed by the number 105, plus the unit, pounds. Notice that to say "Charlie weighs 105" is not enough, for you do not know whether this means 105 pounds, 105 tons, 105 ounces, or 105 what. There are, of course, circumstances where everybody knows what units you mean and it is unnecessary to name them. If Charlie steps on a penny weighing machine in the United States or Canada and gets a card reading "105", he knows that by custom it means "105 pounds". The same Charlie would get a card reading "7 and 7" in England, however, and one reading "47.6" in France. Do you know why?

Here is a repetition to help you remember: Nearly every quantity is a number or a number plus a unit. Most quantities are of the second kind, requiring a number and a unit. In fact, the only quantities that are numbers are numbers themselves! -- like six, two-and-a-half, or 3.7. In any other quantity the unit must be expressed. If you say "The line is 6 long", will anyone know what you mean? Is it six inches, six feet, six centimeters, or six miles? Length is a quantity that must have units attached or it is without meaning. On the other hand, "I drew six lines" is perfectly correct, for the quantity of number (or count) needs no units.

There are some quantities that cannot be expressed by even a number and a unit. These are more complicated and need to be expressed by a number, a unit, and something else. You will get to such quantities later. Don't worry about them.now.

Physical science is inseparably concerned with quantities and relationships among them.

2. Units of Measurement

We have been talking about measurements and comparisons, but has it occured to you that a measurement is a comparison? When you say "My desk is six feet long", you really mean this: "My desk is longer than a one-foot ruler. If I take a footrule and lay off one foot at a time along the edge of my desk, I find that I can lay it off exactly six times." In other words, saying "My desk is six feet long" means exactly the same as "My desk is six times as long as a one-foot unit."

Also, to say "Charlie weighs 105 pounds" means "Charlie weighs 105 times as much as a one-pound weight unit. When you say "My time for the hundred-yard dash is 12.3 seconds," you mean that it takes you 12.3 times as long as a one-second unit to dash a hundred yards. When you speak of a 5-quart jug, you mean the jug holds 5 times as much as a one-quart unit.

Make up some other quantities and then make up similar statements about what they mean. In doing so, there are two things you will have to be careful about.

First, notice that certain units -- like foot, pound, and second -- are "primary" units. They are not derived from anything else. The first person to decide how long "one foot" should be had complete freedom to make it anything he pleased. He could just make two marks on a sheet of paper and say "This is a foot, and everyone will have to agree with me". No one could say he was wrong, because he invented it. The definition of one foot for legal purposes in those countries that use the foot is made in just this way. It is not defined by pencil marks on a sheet of paper, of course, but by scratches on a bar of metal. Do you see why scratches on a bar of metal would be better than pencil on paper? The scratches are so fine that you need a microscope to see them. Do you see why fine scratches are better than coarse ones that can be seen without help? The bar is kept in a safe place so that no one can tamper with it. Everyone then agrees to abide by the law and so "one foot" means the same thing to everybody. Do you see why it is important that everyone agrees on exactly how long a foot should be?

There are certain qualities like volume and area, however, where the story is a little different. You can do two things. You can say that area is really closely related to length; this is what you do when you say that your desk top has an area of six square feet because it measures 2 feet by 3 feet. In the same way volume is also closely related to length. You recognize this when you say that a box measuring 2 by 3 by 4 feet has a volume of 2 x 3 x 4 or 24 cubic feet. When you do this you simply say that the unit you will use to express quantities of area is the area contained in the square that measures one foot each way. The unit of volume is the volume contained in a cube that measures one foot each way. This is the sensible way to do it.

You can also do a much less sensible thing. You can say "I have a perfect right to make up my own volume unit. I will call it a 'gallon' and it will be so big". This sounds like a silly thing to do when you have a ready-made unit in the cubic foot; but that's what the English system of units does, and, of course, we have become used to it. To use the gallon as the unit of volume and the foot as the unit of length means that we must define two units. To use the cubic foot as the unit of volume and the foot as the unit of length means that you need define only one unit.

So you see, when you say that a certain tank holds five gallons, you mean that it holds five times as much as a one-gallon unit. When you say that a certain tank holds five cubic feet you mean that it holds five times as much as a cube measuring one foot each way. Units like the square foot and the cubic foot, which are really derived from other, already-defined units, are called derived units. To keep things as simple as possible, it is always better to use derived units (like the cubic foot) than to use a primary unit (like the gallon). Scientists usually use derived units when they can because it is simpler to do so.

A little more complicated derived unit is the unit of speed, say the mile per hour. You could say that a speed of 30 miles-per-hour means a speed thirty times as great as a unit speed of one mile-per-hour, and to do so would be correct. But what is the unit, mile-per-hour? Is it a unit like the gallon that someone just invented; or is it really a derived unit? A little thought will show you that it is a derived unit, meaning a speed equal to the speed you would have to make to go one mile-unit in a time of one hour-unit. Therefore it is more simple to think of a speed of 30 miles per hour as a speed such that you could cover 30 mile-units in one hour unit.

A better thing to do with speeds, however, is to use the foot that has already been defined, instead of bringing in the new and unnecessary unit of the mile. They are both distances, and only one of them is needed. A speed of thirty miles per hour is the sampe as a speed of 44 feet per second. Can you show that they are equal? Using the foot-per-second as a derived unit of speed is more simple than using the mile-per-hour, of course. Once the foot and second are defined, why bother to define two new units, the mile and the hour?

The other thing you will have to be careful about is in dealing with what are called "irrational" units. The only one you are likely to meet is the degree of temperature. The scale of temperature (both Fahrenheit and Centigrade) are irrational simply because "zero" on the scale does not really mean zero. When you say that the thickness of a shadow is zero inches you mean it has no thickness at all. When you say that an empty candy-box contains zero pounds of candy, you mean it contains no candy at all. But when you say that the outdoor temperature is zero degrees, you don't mean that the outdoors has no temperature at all. For you know that you can have a temperature of 5 below zero, which would then mean "less than no temperature at all". The question then comes up, what temperature means no temperature at all? This temperature, which is what really ought to be called "zero", leads to another means of measuring temperature which is not irrational. You may meet "absolute temperature" later in your study of science. Meantime, notice that saying "This water has a temperature of 50 degrees" does not mean that it is 50 times as hot as something with a temperature of one degree. This strange part of temperature measurement will not concern us in the present study.

3. Making Measurements

You have seen that making a measurement is really nothing more than comparing an unknown with a unit. The main idea in making a measurement is then to have at hand an example of the unit to be used and an instrument for comparing it to your unknown. Often the instrument and the unit are combined into a single gadget, as with the foot rule. Sometimes they are not combined, as in the scales on which you have to put separate weights. We will at the present time talk only about making measurements of distance using the ruler.

If we are going to make scientific measurements, however, we might as well use the same units as scientists use. Although in this country we commonly use the foot and the inch as units of distance, civilized people in most of the world and scientists all over the world use the centimeter.

Get a centimeter ruler and examine it. It will look much like the sketch below. Notice on the sketch, which is drawn life-size, which are the numbered centimeter marks -- the longest lines. Each centimeter spacing is divided in half by a shorter line. Each half-centimeter is divided by four short lines into five parts. Each of these tiniest parts, having a length about the thickness of a dime, is a tenth of a centimeter. Do you see why?

The quantity expressing the distance between two successive longest lines is 1 cm; the half of this distance is the quantity 0.5 cm; and the smallest interval is 0.1 cm. The smallest distances are also called millimeters, though we will have no use for that name.

Now take a pencil or other object and measure its length. To do this, place one end of the pencil exactly opposite the zero-end of the scale and let the pencil lie along the scale with its other end falling wherever it will. Suppose that the other end falls between the 18 and the 19 cm. marks. You then know that the pencil is more than 18 but less than 19 cm long. The smallest marks will help you tell just where between 18 and 19 the length lies. If the end of the pencil lies right on the middle-sized mark lying halfway between 18 and 19, you would record the length as 18.5 cm. If it lies on the second short line between 18 and 18.5, you would record it as 18.2. If on the third line past the middle, as 18.8. Etc. If the end of the pencil does not fall exactly on any of the lines, you take the nearest one. It will almost always be true that the end of the pencil will not fall exactly on any line. You will therefore almost always have to judge which line to choose as the nearest one.

If the nearest line happens to be one of the main centimeter marks, like 18 or 19, you should record the length as 18.0 or 19.0. Be sure you always write the "point-zero" when the length is an exact whole number of centimeters. You will later see the reason for insisting on being fussy about this.

It is a curious thing that nearly every physical measurement you make is in the end made by reading a position on a scale. When you read a thermometer, you really read the temperature the same way you read a ruler. When you read the speedometer on your family's car or the time on a clock, you are really reading a position on a scale, aren't you? This is why it is so important to learn how to read a ruler properly.

4. Significant Figures

When you measured the length of your pencil, you probably found that the end of the pencil fell between two of the finest marks on the ruler, say between 18.6 and 18.7. Suppose that it lay closer to 18.7, though, so that you recorded the length as 18.7 cm. When you did this, you might have said to yourself "I can see that the length of the pencil is really somewhere between 18.6 cm and 18.7 cm. Maybe it is really 18.68 cm but my eyes are not good enough and the ruler is not divided finely enough for me to tell. Anyway, I don't need to know the pencil length that accurately, so I will just call it 18.7 cm, which is the line on the ruler nearest to the end of the pencil."

You have therefore read the length of the pencil to the nearest tenth of a centimeter. Maybe the length of the pencil is a little more than 18.7 cm or a little less. But 18.7 is the nearest tenth of a centimeter. You record the length as 18.7 cm. If your friend asks you "How long is your pencil?", you will tell him "My pencil is 18.7 cm long."

Now suppose your friend tells you that his pencil is 18.7 cm long. What will go through your mind when he tells you so? You might think like this: "He said his pencil is 18.7 cm long. He must have used a ruler to measure it because he gave me the length accurately instead of saying that it was about 18 or 19 cm long. On the other hand, he must not have measured it with a very finely divided ruler and a magnifier, because he didn't say his pencil was 18.72 or 18.727 cm long. He measured it only to the nearest tenth of a centimeter and gave me the result of that measurement."

In other words, when you say your pencil is 18.7 cm long, the number 18.7 really tells two things:

(1) It tells how long the pencil is.
(2) It tells the person who is listening how accurately you measured it.

If you had measured less accurately than to the nearest tenth of a centimeter, you would perhaps say that your pencil is 19 cm long. If you had measured it more accurately, you would have said 18.72, or 18.723, or even 18.7231 cm. (To measure something so accurately that you could say it is 18.7231 cm long, you would have to measure it to the nearest 0.0001 cm. To do this you would need a very special ruler and a microscope to use it.)

Remember then that the quantity that results from a measurement always tells the person who sees it or hears it how accurately the measurement was made. It always means that the last figure is only the nearest figure, and not that it is exactly that figure. Now you can see why we are fussy and insist that you write 18.0 instead of just 18 if your pencil happened to have a length that fell nearest to the 18 cm mark on the ruler. If you say that the pencil is 18 cm long, it would mean that you measured it only to the nearest centimeter. If you took more care and measured it the nearest tenth of a centimeter, then you should be proud of your extra effort and say so by reporting the length as 18.0 cm. In fact, if you took very great care using a special ruler and microsope, you might report the length of your pencil as 18.0000 cm. This means that, even measuring to the nearest 0.0001 cm, the nearest mark was the main centimeter mark at 18.

Here is a list of quantities that might have been measured by some instrument or another. Tell how accurately the measurer must have been working in each case:

A metal rod is 37.17 cm long
A ball bearing weighs 3.267 grams
A bobsled completes its run in 57.07 seconds
A bottle holds 14.0724 cc of water

Now let's think about measuring the length of that pencil once again. Say that you found the end of the pencil to lie between 18.6 and 18.7 cm. You judged it to lie closer to 18.6, so you reported its length as 18.6 cm. In this case you realize right away that your report is only approximate. It isn't exactly 18.6 cm, but only nearer to 18.6 than to any other tenth of a centimeter.

But suppose that the end of the pencil appeared to lie exactly on the 18.6 line. If someone now came along with a magnifying glass and looked at your pencil and ruler, he might say "Oh, no. Look. The end of the pencil lies just a little bit past 18.6. Let's get a ruler where the division of tenths of a centimeter (the finest ones on your ruler) are themselves divided into tenths, and measure the pencil again". What would be the distance between the finest divisions on this super-ruler? Look at your ruler divided into tenths of a centimeter and try to imagine how close together the divisions would be if you had a super-ruler divided into hundredths of a centimeter.

Suppose with this super-ruler you found the length to be 18.62 cm. Would this be exactly correct? Probably not, because someone might come along with a microscope and a super-super-ruler and show that the length is really 18.623 cm. But you couldn't be sure that this reading is exactly correct either, could you? Can you explain why not?

It is important that you realize that any quantity that is measured is never known to be exactly correct. The best you can do is to say that the real value lies closer to some certain mark on a scale than to the next markings before or after it. You never know whether someone else might come along and use a better measuring instrument than you did to get a more accurate value than yours. So you always report any measured quantity like this: Use as many figures as will make the last figure the one selected as "nearest" to some mark.

This rule works both ways. You must be careful never to use either too many figures or too few. Suppose you measure your pencil to the nearest tenth of a centimeter and find it to be nearest to 18.2 cm. You would be unfair to yourself to report it as 18 cm because you really did better than that. But you will be bragging unfairly, if you report the length as 18.20 cm when you didn't measure it to the nearest hundredth of a centimeter but only to the nearest tenth.

When a measured quantity is properly expressed so that the last digit is the "nearest" one, then the figures used are called significant. For instance the quantity 18 cm has two significant figures; the quantity 4.79 grams has three significant figures; and the quantity 18.0000 cm has six significant figures. Tell how many significant figures there are in each of the following quantities.

A length of 17.22 centimeters
A weight of 19.1765 grams
A volume of 180.60 cubic feet
A speed of 17.3 miles per hour
A time of 4500 seconds
A thickness of 0.0012 inch

The last two of these are difficult and you may need your teacher to explain them to you.

Now you are ready to do Experiment 1 in your laboratory manual.

After you have finished the experiment, here are some questions to discuss in class.

Points to Discuss in Class

How many places to the right of the decimal point are significant in these measurements? How many in the averages?

Did everyone obtain the same quantity when one and the same stick was measured by several people? If not, whose measurement was the correct one? Why did different people get different results?

If you had used a ruler divided more finely and a magnifying glass to read it, would everyone have got the same result?

Does it make sense to speak of "the exact length" of a stick?

Does it make sense to speak of the measured length as a quantity that everyone agrees on? Suppose that you own a company that makes and sells gold wire. I mail you an order for 37.5 cm of gold wire of a certain size and you mail the wire back to me. We have never at any time met face-to-face to measure the wire together. When the wire arrives, I measure it to see whether I've been cheated. Is there any reason to suppose that my idea of what 37.5 cm should be will agree with what you think 37.5 cm should be? Why? How closely will be agree?

The whole of physical science rests on a faith in this belief: That when two people make separate measurements of the same quantity, if neither of them makes a mistake, the measurements will agree. How closely will they agree?

5. Doing Arithmetic with Measured Quantities

Suppose you have measured two pencils separately and you wanted to know how long the combination would be if you placed them in line end to end. You would compute the total length by adding the two measured lengths together, wouldn't you? If one pencil was 16.7 cm long and the other 17.2 cm long, the total length would be what?

But now suppose that you measured one pencil with an ordinary ruler in the ordinary way and found its length to be 16.7 cm. Then you measured the other pencil with a microscope-and-ruler arrangement and reported its length as 17.232 cm. What would you report as the total length? From what you have learned in arithmetic, you might be tempted to set down the two quantities and add them like this:

$$\begin{array}{l} 16.7\text{ cm} \\ \underline{17.232\text{ cm}} \\ 33.932\text{ cm} \end{array} \qquad \text{which means the same as} \qquad \begin{array}{l} 16.700 \\ \underline{17.232} \\ 33.932 \end{array}$$

You might report the sum as 33.932 cm, but this would be improper. Let's examine what we have done to see what is wrong about it and what we should have done.

You remember that a quantity reported as 16.7 cm means that the last figure, the 7, was intended to mean that the end of the pencil did not fall exactly on the 7 but closer to 7 than to any other mark. The true length might have been 16.694, for instance, or 16.721. Because we didn't measure it that accurately, we simply do not know what we would have got if we had made the measurement to 5 significant figures. Since we don't know what the next two figures past 16.7 would be, we might write 16.7XX cm as the length. We are pretending that the length is written with 5 significant figures, but we are admitting that we don't really know what the last two are by putting X's for them. We certainly do not, at any rate, know that the next two figures are zeros, as the above addition seems to suppose.

Now, if we try to add the two quantities, we might set the addition down like this:

$$\begin{array}{l} 16.7\text{XX cm} \\ \underline{17.232}\text{ cm} \\ 33.9\text{XX cm} \end{array}$$

and think as follows. In the units column all the way to the right, "X plus 2" is how much? You don't know, so you write down X. "X plus 3" is how much? Again you don't know: write another X in the sum. "7 plus 2" you do know, so you write a 9 in the sum and then complete the addition in the usual way. The result is 33.9XX.

This means 33.9 with some more figures that we don't know. But this means the same as 33.9. Therefore the sum of the two quantities 16.7 and 17.232 is 33.9. You are not entitled to any more than three significant figures and have no right to report more in the sum.

When you learned how to add decimal numbers in arithmetic, you were probably told that 16.7 plus 17.232 is 33.932, as we got first above. Are you now being told that what you learned in arithmetic is wrong? No, you are not, though it might look that way at first. The difference is that in arithmetic you are asked to add two numbers, one of which is *exactly* 16.7 and the other *exactly* 17.232. But *exactly* 16.7 means 16.7000 with as many zeros as you wish, and similarly with 17.232. The sum of these, of course, is exactly 33.932 with as many zeros added on as you wish. But the *measured* quantity 16.7 cm does not mean exactly 16.7 cm, *but only* means "some number of centimeters closer to 16.7 than to 16.6 or 16.8". This uncertainty in the number beginning with the second decimal place creates an uncertainty (remember the "X + 3" in the addition above!) in the sum in the second decimal place. You therefore have no right to report 33.932 as the sum when all the figures after the 9 are uncertain. This uncertainty does not occur if 16.7 means exactly 16.7, as it might well mean if it is not a measured quantity.

The general rule in adding measured quantities is now very simple. If you want to find the sum of 84.62 grams, 171.4 grams, and 42.119 grams, you set them down in the usual way with the decimal points lined up.

$$\begin{array}{r|l} 84.6 & 2\ \text{gm} \\ 171.4 & \ \ \text{gm} \\ \underline{42.1} & \underline{19}\ \text{gm} \end{array}$$

Now draw or imagine a vertical line to the right of the number known with the fewest decimal places. In this example, the "poorest" number is 171.4, because it is known only to a tenth of a gram; all the others are known to better than a tenth. Hence we draw the line to the right of this 4 as in the example above. Then add only the part to the left of the line. Finish the example yourself, and don't forget to add the word "grams" when you read the sum!

Now you are ready for Experiment 2.

When you have completed this experiment, you should discuss the following matters in class.

Points to Discuss in Class

Is it true that you can obtain the combined length of the three sticks by adding the numbers representing the individual lengths? Answer this question by comparing the combined length you obtained by measuring, with the combined length obtained by adding. Remember that two measured quantities can be said to "agree" if they differ by only one or two in the last significant figure.

Does it seem "only common sense" to you that the total length of two or more sticks can be obtained by adding the numbers representing the individual lengths? If it does seem obvious, here are some things to think about that may make you less certain:

(1) What is the combined length of three sticks, one of which measures 2 feet, one 7 inches, and one 4 centimeters? Here is a case where you cannot add the numbers to get the total length. What must be true of the units in each quantity before you may add them?

(2) You may be thinking this way: If I put together a pile of 17 toothpicks and a pile of 12 toothpicks, the two piles together will total 29 toothpicks. If John weighs 80 pounds and Sam weighs 90 pounds, the two together will weigh 170 pounds. If I walk 40 steps, stop, and then walk 50 steps more in the same direction, the two walks together will place me 90 steps from where I started. If Mr. Brown's farm is 5 acres and Mr. Smith's farm next to it is 6 acres, the two farms together will cover 11 acres. If I pour 2 gallons of cider into an empty barrel and then pour in another 4 gallons, the two portions together will put 6 gallons in the barrel. In fact, it often happens that putting two quantities together gives a result that agrees with adding the numbers. We may get into the habit of thinking that "together" means "add". But think of these questions:

If one ocean liner can cross the Atlantic in four days and another can do it in five days, will it take them nine days to do it together?

If I can paint a fence in 15 hours and you can paint it in 10, will it take us 25 hours to do it together?

If I have a glass of water at 75° temperature (about ordinary room temperature) and another glass of water at 100° (about body temperature) will pouring them together give me water at 175° (almost boiling)?

If I walk 40 steps, stop, then walk 50 steps more, must I end up 90 steps from where I started?

You realize, of course, that the answer to every one of these questions is "No". Yet each question asks for the result when two quantities are put together. Do you agree that sometimes the word "together" does not tell you to add? All right, then, what right do you have to say that putting two sticks together permits you to add the numbers representing their lengths to get the total length? In other words, how can you tell when "together" means "add" and when it does not?

The answer to this question is deeper than you might think. But in the end it amounts to this: The only right you have to do so is that experience (that is, experiments such as you just performed) show that you always get the same result whether you measure the total length or add the individual lengths. Your experiment showed that you may do this under two conditions: first, the units must be the same; and second, the two lengths must be along the same straight line.

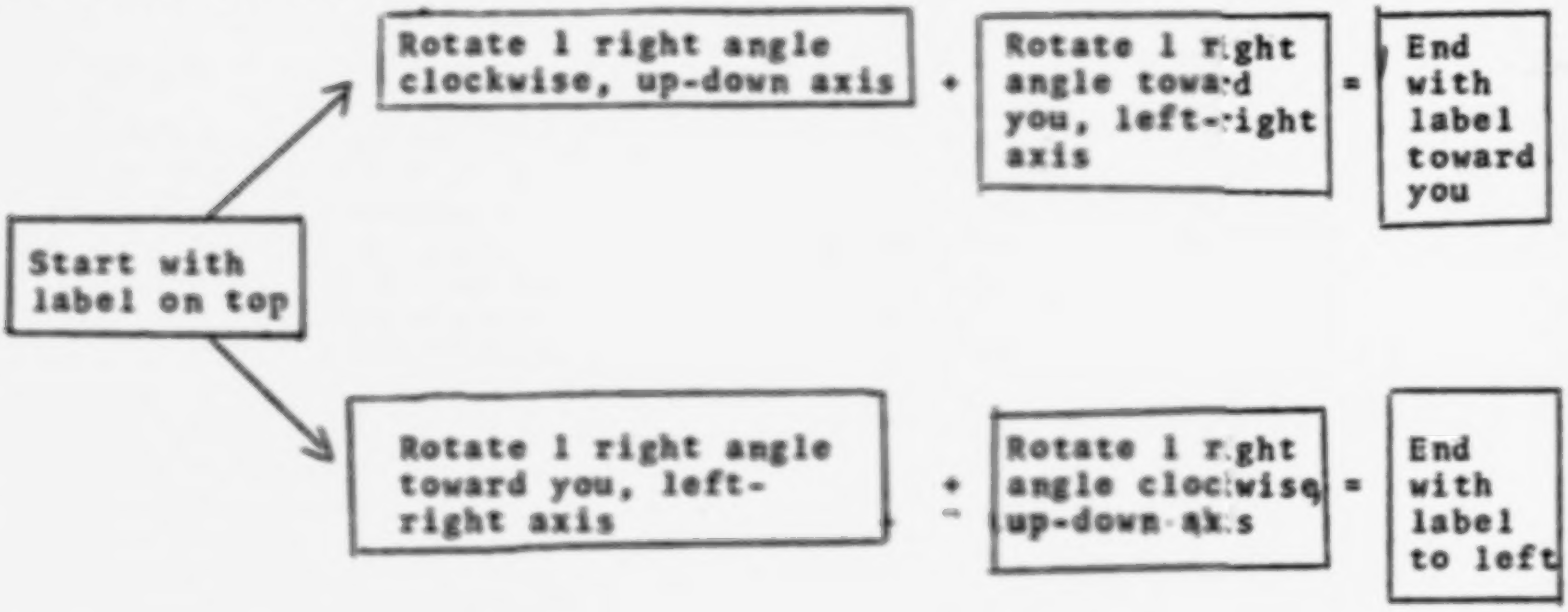
6. Commutativity under Addition

Did you get the same total length of the three sticks regardless of the order in which you lined them up? Does this surprise you? Do you have a right automatically to suppose that adding three lengths will always give you the same result for the combined length no matter in what order you add them? You already know that you may add numbers in any order you wish and always get the same result. This property of numbers is called "commutativity under addition." You have shown by experiment that lengths are also commutative under addition. Not all quantities are commutative under addition. For some quantities, you get a different result when you add A + B from what you get when you add B + A.

Would you like to see an example of a quantity which does not commute under addition? Get an ordinary matchbox and six common straight pins. Stick one of the pins in the center of the top of the box and another in the center of the bottom. You now have an axis around which you can spin the box. You can turn the box around this axis through any angle you wish. Hold the axis vertical in front of you with the label on top facing you so you can read it. Now turn the box around the vertical axis clockwise (in the direction in which the hands of a clock turn) through 1 right angle. The top of the box is still on top, but now a person would have to stand on your left to read the label. Now turn the box further through 2 right angles. The label is still on top, but now a person would have to stand on you right to read it. That is, if you add 1 right angle turn + 2 right angle turns) you put the box in a position with the label on top facing so that a person would have to stand on your right to read it. Now return the matchbox to its original position with the label on top facing you so you can read it. Then rotate the box again clockwise around the vertical axis; but this time turn it first through 2 right angles (the box now has its label still on top but a person must stand in front of you to read it) and then further through 1 right angle. The box now has its label on top, but to read the label a person must stand on your right. That is, if you add (2 right angle turns + 1 right angle turn) you put the box in a position with the label on top facing so that a person would have to stand on your right to read it. This is the same as before. You see then that when you add rotations, they do commute if the two rotations are around the same axis.

But if the rotations are not around the same axis, they do not commute. To see this, stick a pin in the center of each end of the box and also in the center of each side. You now have six pins forming three axes. Hold the matchbox in front of you, top up, and turned so that you can read the label. Hold it by the pins in top and bottom and rotate it around the up-down axis 1 right angle clockwise as before. The label is still on top. Now hold the box by the pins stuck in the sides of the box. These pins make an axis pointing right and left. Rotate the box around the left-right axis through 1 right angle toward you. The label now faces you. Therefore, starting with the label on top and faced so you can read it, 1 right angle turn clockwise around an up-down axis plus 1 right angle turn away from you around a left-right axis leaves the box with the label facing you.

Now perform the two rotations in the other order. Start with the box label up and so you can read it. The pins stuck in the ends of the box form a left-right axis. Hold it by these pins and rotate the box through 1 right angle toward you. The label is now facing you. Then hold the pins stuck in the sides of the box -- the up-down axis -- and rotate the box 1 right angle clockwise. The label now faces to your left. Therefore, starting with the label on top and faced so you can read it, 1 right angle turn toward you around a left-right axis plus 1 right angle turn clockwise around an up-down axis leaves the box with the label facing to your left. Here are the two trials in a diagram form:


Start with label on top
Rotate 1 right angle clockwise, up-down axis
+
Rotate 1 right angle toward you, left-right axis
=
End with label toward you
Rotate 1 right angle toward you, left-right axis
+
Rotate 1 right angle clockwise, up-down axis
=
End with label to left

So you see, adding these two rotations in one order gives you a result different from what you get if you add them in the other order. Rotations around different axes do not always commute.

7. Significant Figures in Multiplying

You now know how to add measured quantities and how to deal properly with their significant figures. Much the same thing happens when you multiply measured quantities. Suppose you have a rectangular card whose width is 23.6 cm and whose length is 37.4 cm. What is the area of the card?

You will remember that you find the area of a rectangle by multiplying the length times the width. You would ordinarily do it this way:

$$\begin{array}{rl}
37.4 & \text{cm} \\
\underline{23.6} & \text{cm} \\
2244 & \\
1122 & \\
\underline{748} & \\
882.64 & \text{cm}
\end{array}$$

You would report the area as 882.64 square centimeters. But by this time you are probably suspicious enough to guess that we are going to find fault with this one, too! We are; let's see why.

You know that "37.4 cm" means a quantity that was measured only to the nearest tenth of a centimeter: closer to 37.4 than to 37.3 or 37.5. We might write this as 37.4X, pretending that we know it to four figures but at the same time admitting that we don't really know the fourth one. In the same way, we will write 23.6X for the other number. Now let's multiply them:

```
   37.4X
   23.6X
 ----------
   XXXX
 2244X
1122X
748X
-----------
883. XXXX
```

Here is the way you perform this strange-looking multiplication: First, you multiply X times 37.4X. How much is it? You haven't the slightest idea, so you might as well admit it and write a string of X's in the first row. Next you multiply 6 times 37.4X. You say, "6 times X I don't know", and you write X all the way to the right in the second row. Then, "6 times 4 is 24", write the 4, carry 2, and complete the second line in the usual way. Following the same method, show how the third and fourth lines were obtained.

Now to complete the multiplication, you start as usual all the way to the right. You bring down the unknown X to the bottom line. In the second column from the right, you say, "X plus X is unknown", and write X at the bottom. In the next colum, you say, "X plus 4 plus X is unknown", and write X in the bottom line again. In the next column, you say, "X plus 4 plus 2 plus X, I don't know", so you write another X at the bottom. This time, though, you say to yourself, "I don't know exactly how much is "X plus 4 plus 2 plus X", but it almost certainly is at least 10, because 4 plus 2 is already 6, and two more digits added to it will probably reach ten or more." So you write the X at the bottom, because you don't know what the sum is exactly, but you carry a "one" into the next column because you're pretty sure it's at least 10. In the next column, the carried-over 1 plus 2 plus 2 plus 8 is 13. Write the 3 carry the 1, and complete the addition in the usual way. There are two "decimal places" in the first number and two in the second. You therefore point off four decimal places from the right in the result and get 883.XXXX. This means that you do not know any significant figures past 883. and therefore you should not report the product any more accurately than 883 square centimeters.

The rule is easy: When you multiply two quantities, the product should contain no more significant figures than are contained in the multiplier with the fewer significant figures. Sometimes it is permissible to take one more figure than the rule allows.

It is not recommended that you multiply by the X-method above. Multiply in the usual way as in the first multiplication above, but when you are finished, "round off" the answer to as many figures as the rule says you are entitled to. Notice that rounding off the first result of 882.64 to three figures gives 883, the same as the X-method shows you are entitled to say.

Now you are ready to begin Experiment 3. When you are finished, you should discuss in class the questions below.

Points to Discuss in Class

What is meant by a scale drawing? Mention some examples of scale drawings that you have seen used in business or school or elsewhere. Does a scale drawing have to be smaller than the object it represents?

You hold a circular card behind your back and ask me to make a scale drawing of your circle without seeing it. Since I have never seen your circle, the best I can do is draw a circle of any size I please on my paper. Will my circle be a scale drawing of yours?

You hold a rectangular card behind your back and ask me to make a scale drawing of your rectangle without seeing it. I draw any old rectangle on my paper. Will my rectangle be a scale drawing of yours?

Any circle is a scale drawing of any other. But any rectangle is not a scale drawing of any other. The reason for this is that it takes only one quantity to describe a circle completely -- its radius. But it takes two quantities to describe a rectangle completely. How many quantities are needed to describe completely the special kind of rectangle called a square? Is any square a scale drawing of any other? It takes three quantities to describe a triangle completely and this is part of the reason why it is a little harder to make a scale drawing of an irregular triangle. Can you name some other shapes for which only one quantity need be given to describe it? Can you name some for which more than one quantity must be given?

You can make a scale drawing only for a flat shape. Flat shapes are called "two dimensional". A body that has thickness or that sticks out above or below the flat is called "three-dimensional". A spoon, a sphere, a rectangular box, and a cylinder are three-dimensional. What corresponds in three dimensions to a scale drawing in two dimensions is called a scale model. Name some three-dimensional objects which require only one quantity to be given to enable a person to make a scale model. Name some that need more than one.

How many areas does a given rectangle have? Only one, of course. Suppose two people compute the area of a given rectangle and get two different answers. Can they both be right? Then if there are two different methods for computing the area of a rectangle, they can both be correct only if they give the same result. O.K.? Now, the formula says that you compute the area of a rectangle by multiplying the length by the width. But is it definite which side of a rectangle is its length and which the width? Suppose you sat down at one desk and I at another, both to compute the area of a rectangle that we are told measures 12.3 cm by 14.6 cm.

You choose to call the 14.6 cm side the length and 12.3 cm the width. You therefore multiply 14.6 x 12.3 to get the area in cm^2. At my desk, meantime, I choose to call the 12.3 cm side the length and 14.6 the width. I therefore multiply 12.3 x 14.6 to get the area. If we both do it correctly we must both get the same result, because the rectangle has only one area. What guarantee have we that we will get the same result?

Does multiplication of a length by a length to get an area commute? Does multiplication of a number by a number to get a number commute? Suppose that the first of these commuted but the second did not. Could we then compute the area of a rectangle by the rule "multiply one side by the other"?

To get the area of a circle you multiply the square of the radius by the number π. What units does π have? The value of π is 3.14159265358979 to 15 significant figures. Of course, no one ever computes the area of a circle by using this many significant figures for π. You always round it off to as many significant figures as you need. How can you tell how many places to round it to for a particular problem given to you?

A given triangle also has only one area, doesn't it? The formula tells you that the area of a triangle can be computed if you multiply 1/2 times the base times the altitude. But you may choose any side you wish as the base; there are therefore really three different ways to compute the area of a triangle depending on which side you happen to choose as the base. With your triangle did you get the same result no matter which side you selected as the base? What guarantee have you that you always get the same result regardless of the selection?

Notice that the answer to this question is not the same as with the similar question we asked above regarding the area of a rectangle. With a rectangle you are multiplying the same two numbers (length and width) in two different orders; you get the same result because multiplication of two numbers commutes. But in the triangle case there is no question of commutation; you multiply different numbers together (depending on the choice of base) yet you still get the same result. Why? We cannot answer this question here beyond pointing out that experimentally you did get the same result for the triangles you measured. That it is true for all triangles is proved by logic in the study of geometry.

In computing the area of a triangle after having selected a particular side to use as base, does it make any difference whether you multiply half the base times the altitude; or half the altitude times the base; or the altitude times half the base; or multiply the base times the altitude and then take half the product; etc.? (There are six possibilities; what are they all?) Does the multiplying of three numbers together commute?

8. Decimal Estimation

Now let us return once again to that pencil we've been measuring -- the one that we found measured between 18.6 and 18.7 cm. Heretofore, we have recorded the length as, say, 18.6 cm if the end fell closer to 18.6 than to 18.7. But a little thought will show you that you really can do better than this. If the end fell about halfway between, you might say that the length is 18.65 cm. If you judge it to be a little less than halfway, you might record the length as 18.64 or 18.63. If a little more than halfway, you might judge the length to be perhaps 18.66 or 18.67. If the length was only a little past 18.6, you might judge it as 18.61 or 18.62; and if almost 18.7, you might estimate it at 18.68 or 18.69. A great deal of experience has shown that the human eye and brain acting together can readily estimate, with surprising accuracy, tenths of a division on an undivided scale. It takes only a little practice for most people to be able to do this quite reliably.

The practice of reading any scale as though its finest divisions were actually still further divided into tenths is called "decimal estimation". It is customary in all scientific work to read a scale by decimal estimation. This amounts to squeezing out of the scale the very last bit of accuracy it is capable of. Experience has shown that a scale whose finest divisions are in the neighborhood of 1/20 of a centimeter or more apart can be read just as reliably by decimal estimation as by having the finer divisions actually ruled on the scale -- and far more easily because the closely ruled lines make for confusion.

In decimal estimation, the last figure (the one that is obtained by estimating tenths between the finest divisions actually ruled on the scale) is regarded as a significant figure. Even though you "guess at it", remember that experience shows that the guess is just as reliable as if the scale were actually divided into tenths of its smallest divisions.

You are now ready to do Experiment 4. After you are finished, we will have some more questions to discuss.

Points to Discuss in Class

Did everyone get the same result on measuring, say, rod #1? Can you expect that everyone will always get the same result when different people make a certain measurement? Now if the only way you can learn the length of a rod is to measure it, and if different people get somewhat different results when they measure it, how can you ever tell what the "true" length of a certain rod is? The answer, of course, is that you can't. It is worth repeating: no physical measurement is ever known to be exactly correct. No one can ever say "The true length of this stick is so-many centimeters." The length can be known with considerable accuracy if highly refined methods are used to measure it, but it can never be known exactly. There is one exception: a certain rod is known to be exactly one meter long. Do you know what this rod is and why it is an exception?

Notice that even a ruler, no matter how "good" it is, is not an exception? Since someone had to make the ruler, he had to measure where to put the marks. Hence the positions of the marks are not known to be correct, and one cannot say "This ruler is exactly 25 cm long."

Even though we agree that we cannot ever tell anyone the "true" length of a rod, yet we still feel that it ought to be possible to tell him the "best" value we know. If several people measure a rod -- or even if one person measures the same rod several times -- the several measurements will not be all the same. Then which measurement do we select as best? There is no truly logical answer to this question, but there is a general agreement by scientists the world over that there is a reasonable answer to the question as follows: If there is a series of measurements of a single quantity and there is no reason to believe that any of them is more reliable than any other, then the "best" value of the thing measured is the average of the several measurements.

The reason behind this agreement is simple. The idea is that every measurement will probably be a little "wrong." But there will probably be just as many "too-big" measurements (with plus deviations) as there are "too-little" ones (with minus deviations). Usually these deviations will largely cancel each other out, and the average will be pretty close to the "true" value. Let's talk again about significant figures.

9. Averages and Deviations

Suppose that two different people each make the same measurement several times. Say that they are both measuring the length of a rod, and one person's results are these:

18.74 18.72 18.74 18.75 18.74 18.73 cm

As you now know, even the best measurer has to expect that he will not get exactly the same value every time he measures a given rod, even if he is equally careful in every try. He takes the average of his results and reports the "best" length as 18.74 cm.

The other person measures the same rod, also six times, and his results are:

18.74 18.70 18.77 18.73 18.68 18.71 cm

He reports, as the best value, the average of the six values, 18.72 cm.

Now, one person reports 18.72 and the other 18.74 cm as the length of the rod. Which shall we take as the best of all? One way to settle the problem is to take the average of the two reports and call the length 18.73 cm. If we do this, however, we are really saying, "There is nothing to choose between the two reports. They are equally reliable and we will therefore take the average of the two reports as the best value." But wait a minute; are they equally reliable?

You will notice that the first measurer's results run from a low of 18.72 to a high of 18.75 -- a range of 0.03 cm. The second measurer's results range from 18.68 to 18.77 -- a range of 0.09 cm. Now, if you had no other information, which measurer would you regard as the more reliable -- the one whose readings fell in the narrow range of 0.03 or the one whose results scattered out over 0.09 cm?

There is no strictly logical answer to this question either. Let us suppose, however, that the true value lies somewhere between the extreme values obtained by both measurers -- that is, between 18.68 and 18.77. It is obvious then that the first measurer was making smaller errors than the second. If we suppose, for instance, that the true value is 18.73, then the deviations made by the two measurers are:

First:	+0.01	-0.01	+0.01	+0.02	+0.01	0.00
Second:	+0.01	-0.04	+0.04	0.00	-0.05	-0.02

You can see that the second measurer was making larger errors than the first.

We instinctively regard as more reliable the measurer who makes smaller errors. We cannot be sure that the second measurer's average is not better than the first. Maybe it is. Maybe the second measurer does make bigger errors; but maybe also he is wrong on the too-big side as much as he is wrong on the too-little side so that the average is quite good. Maybe the first measurer holds his head a little to one side of where he should, and therefore nearly always gets results that are too small. Or maybe his ruler isn't as good as the second measurer's ruler. But if we have no reason to be suspicious of the accuracy of either measurer, most physical scientists feel that the measurer whose results are less scattered is more reliable.

Now if you look at the deviations listed above for the two measurers, you will probably agree that the second measurer's deviations are more scattered. Notice that we are now getting back to the material discussed in Section 1. We have the comparison--sentence:

Number 2's measurements are more scattered than Number 1's.

We are comparing the quality, "scattering of measurements," and saying that 2's is greater than 1's. Is the quality called "scattering" a quality that can be measured? Or is it a quality like happiness, fun, or niceness of appearance, where we only feel that one may be greater than another?

How do we tell that 2's scattering is greater than 1's? We look at the deviations. We see that some of 2's deviations are less than 1's and some are greater. But on the average 2's are greater. That is, we can average the deviations of 1 and the deviations of 2 and see which has the greater average deviation. Take the deviations shown above and average them for each observer. In doing so, pay no attention to whether the deviations are plus or minus -- just average the numbers. Show that the average deviation of Number is only 0.01 cm while the average deviation of Number 2 is about 0.03 cm. It seems reasonable to regard Number 1's measurements as more reliable because they are more consistent and less scattered than Number 2's. When we say "Number 2's measurements are more scattered than Number 1's," we mean that Number 2's measurements have a greater average deviation than Number 1's. In other words, the average deviation of a set of measurements is a kind of sign showing how reliably the measurements were made. The smaller is the average deviation, the greater is the reliability.

In fact, careful scientific measurements are often reported with the average deviation attached to the report. For instance, measurer Number 1 above might report the length of his rod as "18.72 cm with an average deviation of 0.01." This expression is often written in abbreviated form like this: "18.72 ± 0.01 cm." You read the abbreviated form: "18.72 plus or minus 0.01 cm." It means: "The average of several measurements was 18.72 cm. Some of the measurements were greater than the average (plus) and some were less (minus). The average deviation was 0.01 cm."

Let's try an experiment involving deviations. After you have finished, the class will discuss the following questions.

Points to Discuss in Class

Who was the best guesser of the correct number of balls to place in a dish? If you look at the last two lines of Table I, perhaps you can answer the question. Suppose, for instance, that Sam was one of the guessers and that his average guess over all ten dishes was 22.1 balls; suppose also that Mary Ann's average was 21.6. You might say then that Mary Ann is a better guesser than Sam because her average guess was closer to 20 than was Sam's average. But this may not be true.

It may be, for instance, that Mary Ann's guesses ranged all the way from a low of 5 to a high of 52 and that none of her guesses was any where near 20. Yet her average was quite close. You wouldn't want anyone as likely as this to be wrong to do your guessing for you, would you? On the other hand, it may be true that Sam's guesses averaged a little further from the mark than Mary Ann's; but all his guesses lay between 19 and 23. You may prefer Sam's consistency which averages a little off the mark to Mary Ann's wide scattering which averages closer to the mark than Sam's but is never anywhere near.

Comment on the statement: "If Mark stands with one leg in the freezer at -10°F and one leg in the oven at 150°F, his average temperature is a comfortable 70°."

Comment on this one too: "Peggy is an excellent marksman with bow and arrow. She made one shot that fell fifty feet to the left of the bullseye and another that fell fifty feet to the right. The average for her two shots was right on the button."

The point is that the reliability of a measurement is really composed of two parts: (1) How close is the measurement to the true value? and (2) How consistently can we reproduce nearly the same value over and over again? The first of these is called "accuracy" and the second is called "precision." Accuracy refers to how close a measurement is to the "true" value. Precision refers to the consistency among many measurements of the same quantity. It is perfectly possible for a measurement to have high accuracy and low precision: consider the case of Peggy the marksman above. It is also possible to have high precision and low accuracy. For example, suppose you measured the length of a rod using a ruler graduated in tenths of a centimeter. You measured the length as 18.68 cm with an average deviation of 0.01 cm. Sounds pretty reliable doesn't it, with a very high precision? But someone later notices that the ruler you used was sawed off at one end and starts at 1 cm rather than zero. Then your measurements are all one centimeter off. Though the precision is high, the accuracy is very low.

In making measurements, one strives for high precision and high accuracy. The precision of a measurement is always known, because you can always calculate your average deviation. Usually, however, you can only guess at the accuracy, because usually you don't know the "true" value of a measured quantity.

Suppose you have two round buttons. One is a polished metal button and the other is covered with cloth. You want to measure the diameter of each with as high precision as you can. Using a magnifying glass and a special ruler, you measure the diameter of the metal one as 2.173 cm with an average deviation of 0.002 cm. You try the same method on the cloth-covered button. But when you look at it under the microscope you find the surface very rough with ups-and-downs and particles of lint sticking out as much as 0.02 cm. Does it make sense even to try to measure this button with a precision of 0.002 cm? Think up some other examples of measurements where a precision can be so ridiculously great for the measurement as to be without real meaning. Does the hair on a person's head interfere with measuring his height to the nearest millimeter? Does the fact that a person eats, drinks, sweats, and breathes make it sensible to say that a prize-fighter goes into the ring weighing 184 3/8 pounds? (A very small drink of water weighs an eighth of a pound, and a person looses about one ounce of water by the moisture in his breath every two hours, not counting water that he loses by sweating.)

10. Once Again, Lightly

Physical science is the study of matter and its behavior. Scientists have studied physical science long enough to have learned by experience that the behavior of matter is not haphazard but predictable and logical. Logical reasoning involves close attention to the meanings of words and sentences and often involves mathematics too.

One way in which mathematics arises in physical science is through measurements. A quality that can be measured or counted is called a quantity. A quantity may be a number alone, but it may also be a number with a unit attached. When a person speaks of a quantity other than a number, the unit that goes with it must always be stated (or implied) so the person to whom he speaks will understand.

A measured quantity always, therefore, involves a number. This number shows both the value of the quantity and also how precisely it was measured. The statement, "This rod is 6.75 cm long," not only tells the length of the rod but also tells that the rod was measured to the nearest 0.01 cm and was judged to be closer, probably, to 6.75 cm than to 6.74 or 6.76. If the last digit used in writing a quantity is obtained by estimation (judging that digit to be closer to "right" than the next higher or next lower one), then all the digits used in writing the quantity are called significant.

Attention must be paid to the number of significant figures in measured quantities when arithmetical operations are carried out on them. In adding measured quantites, the decimal points are lined up in the usual way. The numbers to be added are then examined to find which has the fewest significant figures after the decimal point. All the numbers are then rounded off to this many decimal places and the addition then carried out in the usual way. (If preferred, the addition may be carried out without first rounding off, then rounding off the sum to as many decimal places as in the number with fewest significant figures after the decimal point.) The sum of a set of measured quantities has as many significant figures after the decimal point as has that member of the set with the fewest significant figures after the decimal point. The same scheme, of course, applies to subtraction.

When multiplying the numbers in measured quantities, the number of significant figures in the product is equal to the lesser of the number of significant figures in the quantities multiplied. The same rule applies to dividing. The idea behind the rules concerning significant figures is simply that a sum or product or quotient cannot be "better known" than any of the numbers used to calculate the sum or product or quotient.

Just because a quantity involves a number, it does not, therefore, follow that a quantity expressed by a number and a unit is a number. We often find that two quantities can be multiplied together by multiplying their numbers together, but this is not always true. When two or more quantities are added, their units must be the same, and the units of the sum will be the same as the units of the individual quantities. When two quantities are multiplied together, their units need not be the same. The units of the product must then be given a special name, for the product will not have the same units as either of the two quantities multiplied.

From the preceding paragraph, you will recognize that the adding of quantities is a far less complicated matter than multiplying them. You may be wondering "How will I ever be able to tell whether the multiplying of two new quantities that I never met before can be handled by multiplying their numbers; does the order of multiplying matter; and what are the units of the product?" Such questions you need not worry about; they will be answered for each case specifically when they arise.

Every measured quantity has an uncertainty about it, because no measuring method is perfect. There is therefore no answer to the question "What is the exact value of such-and-such a quantity?" if the quantity is a measured one. It is often important to know how much uncertainty is involved in a quantity. The uncertainty is revealed in two ways: One is always used, the other sometimes. The first is the simple matter of significant figures. If a quantity is quoted as 18.72 cm, it immediately notifies you that the measurement is uncertain within 1 in the second decimal place -- within 0.01 cm. Less often, the average deviation is used, too. If a quantity is reported as "18.72 ± 0.03 cm," it means that the measurer tried to estimate to the nearest 0.01 cm -- this much is told you in the "18.72" alone. But it also tells you that the quantity was measured many times and the average is quoted, but the results deviated from the average such that the average of the deviations was 0.03 cm.

The number of significant figures quoted and the average deviation both reveal the precision of the measurement. The more significant figures used, the higher the precision. Of the two measurements, 18.72 cm and 18.723 cm, the latter is more precise because 0.001 cm is a "finer" reading than 0.01 cm. Of the two measurements, 18.72 ± 0.02 cm and 18.72 ± 0.03 cm, the former is more precise because the range of numbers leading to the average is smaller. Precision refers to fineness and consistency of measurement.

Accuracy refers to the closeness of a measured value to the "true value." Since the "true value" may not be known, one cannot always tell how accurate a measurement is. It is entirely possible to have very low accuracy and very high precision. The reverse is also possible but not likely.

Further Classroom Discussion

A bird watching club takes part in the annual Christmas bird census conducted by the Audubon Society. The watchers count 3 vireos, 2 waxwings, a flock of terns estimated as 40, 2 robins, 2 orioles, 6 blackbirds, 3 warblers, 5 doves, and a flock of starlings. There is some argument as to how many starlings there are in the flock. The low estimate was 1000 and the high estimate was 3000. They decide to report it as 2000. They also report the total birds observed as the sum of the individual species, namely 2063. Does this report of total birds seen make sense?

The manufacturer of a cleansing tissue cuts the flimsy paper into sheets of 9 1/2 inches by 8 7/8 inches. He marks on the box that the individual sheets measure 9.500 inches by 8.875 inches. Is this sensible?

A French scientist estimates that a meteor would begin to glow when it comes to within 100 kilometers of the earth's surface. An American newspaper prints the story, but to make things easier for its American readers, converts kilometers to miles. The rewrite man finds in the dictionary that one kilometer is 0.62137 miles. The story then appears saying that the French scientist estimated that the glow would begin at a height of 62.137 miles. What would you have said if you had been the rewrite man?

The average speed winning the "Indianapolis 500" automobile race in 1962 was officially reported as   miles per hour. This speed was obtained by dividing the distance traveled (500 miles) by the time required for the winner to go from start to finish, measured as   hours. Do you think the time was measurable this accurately? To be entitled to six significant figures in the speed, both the distance and the time must be known to six significant figures. Assume that the time really was known this accurately. What about the distance? To know 500 miles to six significant figures means that the distance is known to 0.001 mile. This is about five feet.

A calorie chart for foods says that a medium-sized potato is equivalent to 265 calories. Comment on this rating.

An American scientist builds a sun furnace and estimates that he can obtain a temperature of 4000°C. Our rewrite man above handles this story, too, finding that a temperature of 4000°C is the same as a temperature of 7232°F. He prints that the scientist estimates that he can obtain a temperature of 7232°F. What would you have reported?

Unit II.

Constants, Variables, and Equations

1. Constants and Variables

You now know that a quantity is the numerical measure of any physical quality that can be measured or counted. Remember that most quantities must have units attached before they become meaningful.

Let us now look at two lists of quantities:

First List	Second List
The number of sides in a triangle	The length of any triangle's side
How tall the flagpole at your school is	The length of any piece of pipe
How far you live from your school	How far from one house to another
The diameter of Jack's bicycle wheel	The diameter of any circle
The weight of a certain croquet ball	The weight of any ball
The area of your teacher's desk	The area of any rectangle
The freezing temperature of water	The temperature outdoors

Do you see anything special about the first list that does not apply to the second? The important difference between the two lists is this: Every quantity in the first list remains always the same; each quantity in the second list may change from one value to another. For instance, there are always three sides to a triangle, but the length of a side may be any length at all; your school's flagpole is some particular length, but a pipe may be any length at all; *your* house is always the same distance from school, but you can find two houses that are almost any distance apart that you please; a *particular* croquet ball always has the same weight, but you can find *some* ball that has almost any weight you please.

So you recognize that some quantities have the special property of remaining unchanged in value while other quantities may have any value at all (within limits, perhaps). A quantity is called a *constant* if its value remains fixed during the time you are interested in it. If a quantity may have different values during the time you are concerned with it, the quantity is called a *variable.* Try to list a few constant quantities and a few variable ones that you are familiar with.

You might notice that some particular quantity may under some circumstances be considered a constant and under other circumstances a variable. For instance, suppose you were playing with onmeone on a seesaw. You have carefully positioned yourselves so that the board is exactly balanced and then you begin to teeter. As you know, you can now teeter up and down as long as you feel like it. But if your weight suddenly increased and decreased crazily and unpredictably, you wouldn't be able to have much fun on the seesaw, would you? During the short time you play on a seesaw, your weight and your friend's weight remain constant. But you know very well that, over a period of years as you grow up, your weight steadily increases. For the purpose of seesawing one afternoon, you may properly consider your weight a constant. But over a longer period of time, you would have to consider your weight as a variable quantity.

You will have to prepare yourself to accept a peculiar thing about constants: some of them have always the same value, others have a fixed value only during some particular investigation, but may have another value that stays unchanged during another investigation. Constants that always have the same value are often called absolute constants; the number 6, 1/2, 0.022, and 7.96 are examples of absolute constants. Constants whose values stay fixed during any one investigation (like your weight, for instance), but may change from one investigation to another may be called temporary constants. We will have more to say about temporary constants later on.

Right now we will try to measure a certain absolute constant. You are ready to do Experiment 6. After completing it, we will have some questions to discuss.

Points to Discuss in Class

Did you find that the ratio, diagonal/edge, of a square is always the same, regardless of the size of the square? Does it seem reasonable to you that this ratio would not depend on what color the square is, what it is made of, how thick it is, how heavy it is, who measured it, where or when it was measured, or on anything other than that it is a square? If you answered "yes" to both these questions, you have said that it is a property of being square -- a "pure" property that depends only on being square -- that the ratio of diagonal to edge is always the same. In geometry, it is proved that this is true; you have shown experimentally that it is true, at least for those squares that you measured. If you have not already done so, compute the average of your values and write the average at the bottom of the table.

What units does this ratio have? Suppose that you had measured both edge and diagonal in inches instead of centimeters; would the ratio be different? Try it, by having your teacher draw two or three large squares on the blackboard and making the measurements in inches with a yardstick. If you do this, you will probably find out quickly why the metric system is so much easier to use than the English system.

The ratio is a number, without units. It is a peculiar number in that it is not an integer (that is, a whole number), it cannot be written as a fraction no matter what integers you use in the numerator and denominator, and it cannot be written as a decimal no matter how many places you carry it out. It is very nearly equal to 10/7; a closer value is 17/12; and a still closer value is 99/70. But no fraction involving integers only is exactly right. In decimals, the value is about 1.4142, but no matter how far you carry it out, it is never exactly correct. Well then, if you can't hope to represent it by an ordinary fraction or a decimal, how will you name it? The number is usually named "$\sqrt{2}$," which you read "square root of 2."

This number has very many interesting properties, one of which is the origin of its name, "square root of 2." It is the number which, when multiplied by itself, gives 2. See how close you came to the "correct" value (that is, how nearly exactly square were the "squares" you used and how accurately you measured them) by multiplying your average ratio by itself. Why did your measured value not come out exactly correct?

Since when you multiply this number by itself, you get 2, it follows that 2 is the square of this number, or this number is the square root of 2. Calculate the square root of 2 to four decimal places and compare it with your average value. Your teacher will show you how if you don't already know.

[If you like to play with numbers, here is another interesting property of $\sqrt{2}$. You will notice that $\sqrt{2}$ is not quite $1\frac{1}{2}$, but it is more than $1\frac{1}{3}$. It is the fraction, "1 and one somethingth," where the denominator in "one somethingth" is bigger than 2 but less than 3. How about $1\frac{1}{2\frac{1}{2}}$? Well, $1\frac{1}{2\frac{1}{2}}$ happens to be too small but $1\frac{1}{2\frac{1}{3}}$ is too big. The denominator of the fraction should be "2 and one somethingth," the "one somethingth" being between $\frac{1}{2}$ and $\frac{1}{3}$. How about $1\frac{1}{2\frac{1}{2\frac{1}{2}}}$? This turns out to be a little too large while $1\frac{1}{2\frac{1}{2\frac{1}{3}}}$ is too small. The correct last denominator should be more than 2 but less than 3 - say $2\frac{1}{2}$. The fraction then would be $1\frac{1}{2\frac{1}{2\frac{1}{2\frac{1}{2}}}}$. Now it turns out that this is a little too small, but $1\frac{1}{2\frac{1}{2\frac{1}{2\frac{1}{3}}}}$ is too big. A closer value is $1\frac{1}{2\frac{1}{2\frac{1}{2\frac{1}{2\frac{1}{2}}}}}$, which nevertheless is now a little too large. If you keep on writing this already very messy fraction, always changing the very last 2 to 2 1/2, you keep getting closer and closer to $\sqrt{2}$. Try working it out using, say, six 2's and then seven 2's. The correct value will be between your two results. Your teacher will help you if you get mixed up. You will find a guide for doing the work systematically on page 20 of your workbook.

[You must not get the idea that $\sqrt{3}$ could be calculated by using 3's in placed of the 2's in the continued fraction for $\sqrt{2}$. You can't. Perhaps when you study more mathematics you will learn why $\sqrt{2}$ can be calculated this way.]

2. Arithmetic with Quantities

In Experiment 6 you calculated the ratio of length of diagonal to length of edge for a square. That is, you *divided* the length of the diagonal by the length of the edge. But how could you do this? When you learned how to divide, you were taught only how to divide ~~one~~ *number* by another *number*. Lengths are not numbers -- they are *quantities*, consisting of a number plus a unit. How can you divide things other than numbers? Can you divide an umbrella by a boxcar or a picnic by a jar of olives? Clearly you cannot divide any old thing by any old other thing. When then does dividing one quantity by another mean something?

Notice that it could hardly be an accident that dividing the *number* of the quantity expressing the diagonal length of a square by the *number* of the quantity expressing the edge-length of that square should always give the same result. In other words, in this experiment the dividing of one length by another at least *appeared* to mean something. We can give no strictly logical answer to the question of when you may multiply or divide quantities that are not numbers. But we learn, sometimes by experiment and sometimes otherwise, that certain quantities can be multiplied or divided to give meaningful results.

For instance, you already know that you can multiply the length of a rectangle by its width (both quantities but *not* numbers), and get the area of the rectangle as a result. You know that you can multiply your wages per hour by the length of time you work (again quantities that are not numbers) to get your total pay. These are cases where you can perform arithmetical operations on the numbers appearing in quantities and get meaningful results. But suppose you divide the speed of a motorcycle by the number of buttons on the jacket of its driver; or multiply the weight of a bird by the number of leaves on the tree-branch it's sitting on. You can perform these arithmetical operations, too. But do the results mean anything? One cannot say logically that either of the two last operations is really nonsensical. One cannot, that is formulate a logical rule that will tell you when a certain mathematical operation upon physical quantities is useful and when it is not. One of the important goals of physical science is to seek out those cases where mathematical operations on physical quantities are useful and meaningful.

The case for addition and subtraction is one you are already familiar with. Try to recall Experiment 2, where you added the lengths of some sticks and found the resulting quantity equal to the length of the train of sticks laid down end-to-end. You are also aware that adding the weight of one rock to the weight of another will give you the weight of the pile made of the two rocks together. Now it is clear that the train of sticks has *some* length, and the pile of rocks has *some* weight, and that this length and this weight have meaning even if the individual sticks and rocks are not measured. That is, you don't have to know the individual weights or lengths in order for the total length and total weight to have meaning. You don't even have to have a defined length-unit or weight-unit in order to tell someone the length or weight. (You can tell someone how long the train of sticks is by holding your hands the right distance apart and saying "This long.") All this discussion says that the length of the train and weight of the pile are quantities all by themselves. They are meaningful quantities whether they are thought of as sums of components or not. They are meaningful quantities whether a unit of measurement happens to be handy or not.

A given train of sticks, moreover, has always the same length. Its actual length does not depend on what units you choose to measure it in. You may say the train is 24 inches long, or 2 feet long, or 61 cm long. These are all the same length, for giving the length different names does not give it different values. In the same way, you realize without any need to explain that "my father," "Dad," and "Mr. Brown" all may refer to the same person.

Now let us consider a specific case of adding two stick lengths. A stick 25.4 cm long and another 38.1 cm long will, laid end-to-end, produce a train 63.5 cm long. If you were asked whether these two sticks laid end-to-end would span a distance of 61 cm, you would say "Yes." Now this property of the two sticks of being able to overspan a distance of 61 cm has nothing whatever to do with the fact that you made the measurements in cm. If someone else came along and met the same problem, he might ask "Will these two sticks together span this distance of 24 inches?" (24 inches happens to equal 61 cm, withing the precision of two significant figures.) To find out, he might measure the two sticks, find that they have lengths of 10 inches and 15 inches, notice that the sum of 25 inches is greater than the given 24 inches, and then answer "Yes, they will span the given distance." An uncivilized man who never heard of a ruler and has no concept of arithmetic might arrive at the same conclusion without making any measurements at all. A highly civilized man from outer space may make the necessary measurements in units you never heard of and come to the same conclusion. The point is that the sticks either do or do not span the space. The sticks do not know what means you are going to use to find the answer, and do not change themselves so that they give one answer for one method and another answer for another method. The behavior of the sticks is a property of the physical world, not of the methods that man uses to study the world. Remember this, for it illustrates the most important precept of all of physical science: The behavior of the Universe is independent of the means used to study it. If you have a problem to solve and the answer you get depends upon the method you used to solve it, then you cannot be sure that that answer is right.

But suppose that two different people measured the two sticks, one using inches and the other using centimeters. One stick is 10 inches long and the other is 38.1 cm long. Will the two sticks span a distance of 61 cm? Notice that the problem has not changed; we are still talking about the same two sticks and the same distance to span. By this time, also, we are convinced that one can obtain the total length of two sticks laid end-to-end by adding the quantities representing their individual lengths. Therefore the sum of the quantities "10 inches" and 38.1 cm" must be the quantity representing the total length.

Now if we add 10 + 38.1 we would get 48.1, would see that 48.1 is less than 61, and would conclude that the two sticks would not span the distance. On the other hand, we concluded from earlier discussion that the two sticks do span the distance. Here we have two different answers to the same problem, depending upon how we worked it. That's not allowed! What is the trouble?

If we stick exclusively to cm we find that 25.4 cm plus 38.1 cm is 63.5 cm which exceeds 61 cm, and the sticks do span the distance of 61 cm. If we stick exclusively to inches, we find that 10 inches plus 15 inches is 25 inches which exceeds 24 inches and the sticks do span the given 24 inches. If we stuck exclusively to miles, or feet, or versts (used in Russia), or grixes (used on the planet Nonesta) we would never have trouble: we can find the sum of two quantities, under these conditions, by adding the numbers representing the quantities. When the two quantities use different units, however, you cannot get their sum by adding the numbers representing the quantities.

You must understand that there is nothing whatever wrong with "adding 2 inches and 3 centimeters." This is a quite reasonable and meaningful operation; they do have a sum. The wrong part enters only when you try to add the numbers 2 and 3 and expect the number-sum to represent the length-sum. In other words you can add two lengths together to get a total length whether or not the two lengths are expressed in the same units; for this is a physical operation in which the sticks have no way of knowing what some human being chooses to call their lengths. But as soon as that human being wants to compute their combined length by adding numbers, the quantities must be expressed in the same units. If you wanted to add 10 inches to 38.1 cm to get a single quantity representing the sum, you can do it only if you change 10 inches to centimeters or change 38.1 cm to inches and then added the numbers. You can make this change if you know that one inch is 2.54 cm. Adding weights some of which are in the metric and some in the English system is also possible when you know that one pound is 453.6 grams.

Finally, suppose that you wanted to add 3 gallons and 4 hours, what would you get? Realizing that numerical addition is forbidden unless the quantities are in the same units, you seek first either to change 4 hours to gallons or to change 3 gallons to hours. But this cannot be done, for the gallon is a unit of volume and the hour a unit of time. There can be no way of converting the one to the other because they measure different things. Their sum could not then possibly be a quantity, because a quantity is the measure of a quality, not the measure of two or more qualities. There is no meaning to the sum of two quantities that are measures of different qualities.

All this discussion can be summarized in the following

Rule: Two quantities can be added (or subtracted) numerically if they are measures of the same quality and are expressed in the same units. The sum (or difference) is another measure of the same quality and has the same units.

It is important that you understand where this rule comes from. The rule is not a law that was passed by Congress, or that your teacher or the President of the United States or the Pope decreed you must follow, or that the Bible or some textbook said everyone must obey. The rule comes from a logical examination of the meanings of words and of fundamental principles. It is a rule forced on us by the nature of the world, not forced on you by someone's say-so.

Here is a word of caution. Do not underline the Rule above or box it carefully in red ink as something important to remember. It is important to remember, of course. But if you did not understand the discussion that led up to the rule, you miss the whole point by merely memorizing the rule. If you did understand the discussion, then you know the rule without memorizing it. Memorizing is very unimportant in physical science.

You must work a few examples to make sure you understand the ideas of adding physical quantities.

1. How far would three steel rods stretch if laid end-to-end. One rod is 14 cm long, one is 2.62 cm long, and one is 10.941 cm long?

2. A stack is made of four thin aluminum plates laid flat one on top of another. One plate is 0.0346 inches thick, one is 0.123 cm thick, one is 0.00248 cm thick, and the fourth is 0.001756 inches thick. How thick is the stack?

3. A flask contains 22.71 cc of water. An irregular lump of glass is placed in the flask and the water level rises to 57.22 cc. What is the volume of the lump of glass?

4. A flask contains 34.65 cc of water. An irregular lump of marble weighing 17.212 grams is dropped into the water, so that the total volume is now the sum of the volumes of water and marble. What is the total volume?

5. Three pieces of brass are placed in a box weighing 586 grams. One piece weighs 1.748 pounds and the second weighs 13.42 ounces. The box with all three pieces weighs 2271 grams. What is the weight of the third piece of brass?

3. Multiplying and Dividing Quantities

The logic involved in deciding when you may multiply or divide physical quantities is somewhat simpler than for adding and subtracting. The corresponding rules are therefore a little less restrictive. You will remember the basic restriction. It was that adding two quantities together always means lumping one portion of a certain quality (like length, volume, weight, etc.) together with another portion of the same quality. The nature of addition is such that we can attach meaning to a sum only when we add measures of the same quality. This is not the case with multiplying.

For instance, remember again that you can multiply the length (say in feet) by the width (also in feet) of a rectangle to get its area in square feet. This is a case where you are multiplying two measures of the same quality. But you can calculate the distance a car travels by multiplying its speed by the time it takes to make the trip. Three different qualities (distance, time, and speed) are involved here, all with different units. You can multiply the area of a box-top (square inches) by the height of the box (inches) to get its volume (cubic inches); again three different qualities and three different units being involved.

The questions now are: What quantities may you multiply together numerically? What units may the quantities have? And what units does the product have? The first of these questions must be answered for each particular case and will be touched upon more thoroughly in Section 5 below. In general there are no restrictions at all on what quantities may be multiplied together. (That makes things easy, doesn't it?) The two questions about units are settled by the following

Rule: You may numerically multiply two quantities whether or not they have units or whether or not the units are the same. You attach to the product a new unit whose name is formed by joining the names of the two individual units together with a hyphen, either one first.

There are a few conventions used generally in connection with this rule. For one thing, only the second member of the compound name is made plural. When both quantities have the same unit, the compound unit is usually named by using the word "square" in front of the common unit. Thus the area of a rug measuring three feet by two feet is usually given as "6 square feet," though there is nothing logically or grammatically wrong with calling it "6 foot-feet." If only one of the two quantities has units, the units of the product are the same as the units of this quantity.

You might now protest that we went to a lot of trouble to explain and justify the rule for adding quantities and even scolded the person who wanted to memorize the rule as a substitute for understanding it. Why now do we give this new rule for multiplying without any justification, so that the only way anyone can learn it is to memorize it? You have a right to be given an answer to this question.

You will remember that adding two quantities is very much a common-sense process. You can add two herds of sheep together and obtain the number in the combined herds by adding the numbers for the individual herds. You can do the same with baskets of apples or gallons of cider. Under the proper conditions, you can do the same with lengths of sticks, intervals of time, weights of rocks, etc. Combining two portions of the same quality, as we said before, is almost an intuitive process that yields a larger portion of the same quality; that is, the numerical sum. We saw, however, that conditions have to be proper. If the sticks are not laid end-to-end, if the time intervals are not consecutive, if we add the weight of a half a pile of rocks to the weight of the whole pile -- the numerical sums may not have as much meaning as we might at first think. The truth is that one has to be a little careful even in this intuitively "simple" process.

Multiplication, however, is not so intuitive. There are times when multiplication is merely a compact way of stating a long addition. But there are times (in fact most times) when multiplication is not a kind of addition. In these cases, we have no immediate meaning to attach to the idea of the product of two quantities, and we are free to attach to it whatever meaning we wish that is consistent with experience. Each case of multiplying two physical quantities together has to be examined separately, as we shall see later, in the light of such experience. Since there is no automatic, common-sense, intuitive meaning to the product, there is no automatic name for the units in which the product is measured. We can call it what we want. Most people feel that it is more reasonable to name the unit of the product after the names of the things multiplied together. But there is absolutely no logical reason why this should be done (and in fact it is not always done); it is only convenient to do so. Thus the rule for addition is a logical one and was carefully explained. The rule for multiplication cannot have a logical basis and is only a convenience that has to be learned. The situation here is somewhat similar to, say, your eating habits. You have two rules for eating: you wash you hands before eating and you wash the dishes after eating. Why not the other way around? There is a very good logical reason for washing your hands before the meal, but a much less sound one for washing the dishes after . The one "rule" is logical but the other is largely convenience.

The situation is similar with the process of dividing quantities. Here is the

Rule: You may numerically divide two quantities whether or not they have units or whether or not the units are the same. You attach to the quotient a new unit whose name is formed by writing first the unit of the dividend (numerator), then the word "per", then the unit of the divisor (denominator).

Again, there are conventions to be observed. Frequently in writing, the word "per" is replaced by the diagonal slash "/", just like a fraction bar, which in fact refers to dividing the upper or first unit by the lower or second one. In speech, the slash is read as "per". Only the first unit named (numerator) is made plural. When both quantities have the same unit, the quotient is a quantity without units -- that is, a pure number. (Very often, however, one sees units, like "feet per foot" or "gallon per gallon", used to emphasize the units from which the quotient was derived, though this is not necessary.) When only the numerator has units, the quotient has the same units. When only the denominator has units, the quotient has the same units with the word "per" in front. In this case, the name is always singular.

Notice that the person who says "You cannot multiply or divide one quantity by another unless they are both numbers" is no more (or less) right than the person who says "Oh yes you can." There is obviously nothing to stop you from multiplying the number of one by the number of the other. The important question is: is it worth doing? We answer this question this way: if the product has meaning, it is worth having done it. Physical science is much concerned with discovering when such arithmetical operations on quantities have meaning.

You have discovered that dividing the length of the diagonal of a square by the length of its edge has meaning, and is therefore permissible.

These rules are far less formidable than they may look. You only need a little practice to get the idea. Here are some questions to practice on:

If an automobile can travel 180 miles on 10 gallons of gasoline, what do you get when you divide 180 miles by 10 gallons?

If you can travel 24 miles on your bike in 3 hours, what do you get when you divide 24 miles by 3 hours?

If 55 gallons of paint weighs 495 pounds, what do you get when you divide 495 pounds by 55 gallons?

If 15 1/2 pounds of hamburger costs 620 cents, what do you get when you divide 620 cents by 15 1/2 pounds?

If an airplane needs 1350 gallons of gasoline to travel 135 miles, what is the meaning of the quantity 1350 gallons divided by 135 miles? What is the meaning of 135 miles divided by 1350 gallons?

The glass for a large telescope mirror has to be cooled very slowly. In one case the glass was cooled from 800 degrees to 500 degrees in 30 days. What is the meaning of dividing 300 degrees by 30 days?

A man strings 15 tennis rackets in 5 days. What is the meaning of 15 divided by 5 days?

A garden is 30 feet by 40 feet. What is the quantity 30 feet time 40 feet and what does it mean?

Thirty marbles cost 15 cents. What is the meaning of 30 divided by 15 cents and what is the meaning of 15 cents divided by 30?

An iron pipe 7 feet long weighs 28 pounds. What is the meaning of 7 feet divided by 28 pounds? What is the meaning of 28 pounds divided by 7 feet?

A baseball player makes 72 hits out of 240 times at bat. What are the meaning and the value of the quotient 72 divided by 240? The newspaper reports this batter as having a batting average of "300." Where does this number come from?

If you have not already done Experiment 8, now would be a good time to do it. Then come back and we will have some more.

Points to Discuss in Class

How many significant figures are you entitled to in calculating the ratios of circumference to diameter?

What are the units of the ratio, according to the rule?

Did you find that you got the same ratio (allowing, of course, for a little experimental error) for every circle? Does it seem reasonable to you that this ratio should be independent of what the circle is made of, who made it, how thick it is, where it was made and measured, or anything else other than that it is circular? Table I shows you that it is purely a property of being circular that the ratio of circumference to diameter is always the same -- at least for the circles you measured, and when the measurements are made in centimeters.

Did you find that the ratio changed when you switched to inches for the measurements? When you switched to widgets? Does it matter what units you use to make the measurements?

According to the rule for units of a quotient, when the numerator and denominator both have the same units, the quotient has none; that is, the quotient is a pure number. The quotient, in other words, does not tell you the units of the two numbers divided. Might this be because it isn't necessary to tell, because you get the same result no matter what units are used? At least you have shown experimentally that it doesn't matter for the units you used -- centimeters, inches, and widgets. Did anyone in your class find that he got a different result for his invented unit? What does this show? Notice that the fact that everyone in your class got the same ratio even when very many different invented units were used does not prove that no one will ever invent a unit for which the ratio will be different. This sounds like a hard thing to prove, doesn't it -- that noone will ever find such a unit? Nevertheless we will prove exactly that at the end of this Unit!

The number π is a tremendously important number in mathematics and physical science and elsewhere, too. The fact that it is the ratio of circumference to diameter of a circle is only one of very many places where it pops up. You will see a few more places as we go along. And, it is to be hoped, you will see many more in your future study.

The value of π was given in the discussion following Experiment 3 to fifteen significant figures. Of course, no one ever determined π this accurately by experimental measurement. Even if you used a microscope and super-ruler that could measure to 0.0001 cm, you would have to measure a circle about six times the diameter of the earth to get this accuracy! How then can π be determined so accurately? Simple -- you use some of its other properties.

[If you like to play with numbers, here is another property of π by which you could compute its value to very great accuracy if you had the patience. Multiply 4 x (1/2), and you will get 2.00, which you know is much less than π. In other words, to get π, we would have to multiply 4 by something much larger than 1/2. Very well, we will add something to the 1/2 and try again. The thing we will add is 1/3: try 4 x [1/2 + 1/3] . If you work this out, you will find it comes to about 3.33, which of course is too large.

We overshot the mark, so we will now subtract a little off: try 4 x [1/2 + 1/3 - 1/15]. This comes to 2.93, which now is too small. We overshot the mark again, but we're coming closer! So we add a little back on, this time adding 1/35. We find that 4 x [1/2 + 1/3 - 1/15 + 1/35] is too big, but not very much and we are now closer to π than before. We can get still closer by subtracting a little away again, this time subtracting 1/63. This time we find that 4 x [1/2 + 1/3 - 1/15 + 1/35 - 1/63] is too small, but closer than ever. We must add a little back on. Continuing in this way we keep getting closer and closer to π, one time too large and the next time too small, but always closer than the time before.

[The trick, of course, is to know exactly what next to add or subtract inside the parentheses. Obviously you cannot add or subtract any old thing you please, and expect to get closer to π every time regardless of what you add. (Of course, if we knew the value of π ahead of time, we could always tell what has to be added or subtracted. But remember that we do not know its value beforehand.) There is a special scheme to the series of fractions in the parentheses.

[To learn this scheme, notice first that every fraction has 1 for its numerator. The denominators of the fractions (after the first fraction, 1/2) are 3, 15, 35, 63, etc. Do you notice any pattern in these numbers?Compare this series to the series you would get if you increased each number by one. The new series would be 4, 16, 36, 64, etc. Do you see a pattern now? The new series is simply the squares of the even numbers" in their natural order. The next fraction to be added to the 1/63 last used above is 1/99 (99 is one less than 10 x 10) and the next one after that to be subtracted is 1/143 (12 x 12 = 144). And so on.

[You might want actually to work out π this way. You would get π correct to two decimal places by taking six fractions, and correct to three decimal places by taking sixteen. The worksheet for Experiment 9 will help you to systematize the work. Perhaps if there is a company that uses an electronic computer near your school, your teacher might arrange to have you visit there and have it compute π for you from this series using perhaps 150 fractions in the parentheses. This would give π correct to about 5 decimal places. The machine could do this for you very quickly, whereas it would take you with a pencil and paper many days. Actually this series is a very slow way of computing π, though a very simple one. There are very much faster and better ways to do it, though much more complicated. It is hard to see how the squares of the even numbers could be related in such an elegant way to the ratio of circumference to diameter of a circle. It is one of the beauties of mathematics that such relationships exist and can be proved to be true.]

4. Symbols

The number 12 is a constant, isn't it? But 3 x 4 is also 12, and so is 6 x 2, and so is 24/2, and so is 7 + 5, and so is 17 - 5, and so is $\sqrt{144}$. Since 12 is equal to all these (and many more, of course), is 12 therefore a temporary constant -- or even a variable? The answer is no. You must be careful to distinguish between the value of a number and the name of the number. For instance, "twelve" and "12" are different names for the same thing; in French they call it "douze," in German "zwölf," and in English we also sometimes say "dozen"; -- but they all mean exactly the same thing. In the same way "3 x 4" is still another name for "twelve," and so is "7 + 5," and so is $\sqrt{144}$. They are all merely different ways of writing exactly the same number.

Similarly, consider a certain rock weighing 12 ounces. The weight of this rock is also 3/4 of a pound, or 0. 75 pound, or 341 grams. Does the fact that the weight of the rock has all these values mean that the weight is variable? No. Again "12 ounces," "0. 75 pound," and "341 grams" are simply different ways of saying exactly the same thing -- different languages, if you please. Be very careful to distinguish between a quantity itself and its name. The quantity, if it is fixed, has only one value, but it may have many different names. One kind of name especially convenient to use for a quantity is the sort of name called a "symbol."

When one thing is used to represent another thing, the first thing is called a symbol. You have been using symbols almost all of your life, but you may not realize it. Take for instance a dog. A dog is a certain kind of four-legged animal that barks, wags its tail, and likes to be petted. You refer to this animal orally by pronouncing the word "dog," but you do not have any trouble confusing the animal itself with the sounds you make when you pronounce the word "dog." The sounds are a symbol representing the animal. In the same way, certain marks on a piece of paper -- the marks look like this: d o g -- are a symbol for the animal, but are not the animal itself.

You can see how useful symbols are. Wouldn't it be troublesome if everytime you talked about your dog you could do so only by lifting him up, pointing to him, and saying "ugh"? Imagine how it would be if you had to do that with every thing you talk about. All of your speech is really the use of symbols, and of course it takes a baby a long time to get used to using the same symbols that other people use so they can understand him. In the same way, when you learned to read you had to learn a whole new set of written symbols before you could understand what you were reading.

Physical scientists find it very useful to use symbols in addition to the ordinary ones used in speaking and writing. Most of them are just new and easier names for the quantities they deal with. For instance, you have already seen that when you divide the length of the diagonal of a square by the length of its edge, you always get the same number. We could write the sentence

(Length of diagonal) (divided by) (Length of edge) (always gives) (same constant).

where each separate idea in the sentence is put in its own parentheses.

Now let us rewrite the sentence using symbols. Nouns first: Instead of writing "Length of Diagonal" (which is already a symbol anyway!), we will write simply "D". Instead of writing "Length of Edge", we write "E". We have already used $\sqrt{2}$ as the symbol for this particular "Same constant," and we might as well continue doing so. The sentence now looks like this:

D divided by E always gives $\sqrt{2}$.

Next, we will agree that "always gives" will be symbolized by "=" and "divided by" by "/". The sentence then reads

$$D/E = \sqrt{2}$$

You should not let this strange-looking sentence trouble you. It may look formidable or foreign, true; but perhaps you can remember the day that the sentence you are now reading looked strange and undecipherable. The sentence "$D/E = \sqrt{2}$" is merely written in a foreign language, but it is a language that is very easy to learn.

You have used symbolic statements like this before, of course, but it is important that you understand the exact meaning of such a sentence. (People also call them "equations" or "formulas" but they are really only sentences.) Perhaps the most difficult thing about such a sentence is the meaning of "=". To say that one thing equals another is not always exactly clear in meaning. Fortunately, however, when dealing with quantities, the meaning is quite exact. Two quantities can be equal only if they are quantities having the same magnitude and the same units -- only, in other words, if they are no more than different names for the same quantity. This statement is worth repeating:

In an equation involving physical quantities, the two sides of the equation are merely different names for the same quantity. This means that the whole left-hand side of the equation (no matter how complicated it may look) and the whole right-hand side are different names for the same quantity. Do you see how fundamentally simple an equation is? An equation is nothing more than a sentence that says that one quantity is merely a different name for another quantity; that is, that the "two" quantities are really only one under different names.

In the equation $D/E = \sqrt{2}$ for instance, $\sqrt{2}$ is the name of a certain number which you learned how to work out. Experiment 6 showed you that D/E is another name for this number. For what does "D/E" mean? It means the quantity you get when you divide the variable D by the variable E. But D and E are both lengths. For example, D might be 14.14 cm and E might be 10.00 cm. Then according to rule for dividing quantities, D/E is a quantity whose number is 14.14/10.00 or 1.414. The units of thisnumber may be found by the rule: since D and E both have cm as their units, D/E has no units, or is a pure number. Thus D/E is simply 1.414 -- as is also $\sqrt{2}$ to the accuracy of our measurement. That is, "D/E" and $\sqrt{2}$ are merely different names for the same quantity.

Write an equation like the one above for the relationship between the circumference of a circle and its diameter. Use "C" and "D" as symbols for circumference and diameter. Is "C/D" an other name for π? Does your experiment tell you so? Does dividing C by D give a number without units?

Just to show yourself how easy it is, try your hand at translating the following sentences into symbolic equations. Use whatever you like as symbols for the quantities involved.

1. If you lay two sticks down end-to-end in a straight line, the total length can be found by adding the lengths of the right-hand and left-hand sticks.

2. The average speed of a car during a trip may be found by dividing the distance traveled by the time for the trip.

3. The cost of a pile of golf balls is the price of one ball multiplied by the number of balls in the pile.

4. The radius of a circle is half its diameter.

Also, make up formulas to show the relationship between

5. The perimeter of a rectangle and its length and width.

6. The perimeter of a square and the length of its edge.

7. The weight of a pile of golf balls, the weight of one golf ball, and thenumber of balls in the pile.

8. The cost of a pile of porkchops, the weight of the pile and the cost per pound.

5. Multiplying and Dividing

Of course you know how to multiply two numbers together and how to divide one number by another. In fact you are probably quite skillful at it. Don't let the title appearing at thehead of this paragraph make you think you are going to have to go through all that again. Instead, the present section will try to tell you something about what multiplying and dividing mean. First --- multiplying.

It is easy enough to see what is meant by multiplying two integers together. Integers, you remember, are the numbers you use in counting -- like zero, one, two, seven, forty-three, and one-hundred-twenty-one. But there are also numbers that are not integers but lie between two consecutive integers -- like 6 1/2, 14.712, $\sqrt{2}$, and π. What consecutive integers does each each of these lie between?

If you multiply two integers together -- say 6 and 7 -- the product is defined as that number which you would get if you add 7 and 7 and 7, etc., six times. And it happens, as you know, that adding 6 and 6 and 6, etc., seven times gives you the same result; that is, multiplying of integers is commutative. Since you obviously can do this with any two integers (although with big numbers it may take a long time to do it), there is no trouble with the meaning of multiplying integers. You know, too, that whenever you multiply one integer by another, the result has to be an integer.

Also when you multiply a non-integer by an integer -- say 6 x 7 1/2 -- you say that the product must be 7 1/2 + 7 1/2 + 7 1/2, etc., six times. This, too, you can do for all cases.

The trouble starts when you try to multiply numbers that are not integers; for instance, 7.5 x 6.3. The question here is not "How much is 7.5 x 6.3?" Before we can say how much it is, we must first decide "What does 7.5 x 6.3 mean?" We have agreed on what is meant by multiplying two numbers of which at least one is an integer. We have not yet said what multiplying means when neither is an integer. But there is really an even more basic question than that. One way to phrase the more basic question comes from realizing that "7.5 x 6.3" cannot mean 7.5 + 7.5 + 7.5, etc., 6.3 times. Thus the truth is that "7.5 x 6.3" doesn't mean anything until we say what it means. Since "multiplying" has so far been defined only when at least one of the factors is an integer, we are quite free to make multiplying numbers other than integers mean anything we want. The most basic question then is "What do we want multiplication of two non-integers to mean?"

When we look at the matter this way, we can see that two requirements would be desirable if we could meet them. First, we would like the product of two non-integers to mean something useful -- else why bother to define it in the first place? Second, since we know how much are (7.5 x 6) and (7.5 x 7), we would like (7.5 x 6.3) to lie between (7.5 x 6) and (7.5 x 7) -- simply because 6.3 lies between 6 and 7. And we would like (7.5 x 6.3) and (7.5 x 6.2) and (7.5 x 6.4) to be defined in such a way that (7.5 x 6.3) lies between the other two. In fact, (we're going to use symbols now) if G is any number greater than 6.3 and L is any number less than 6.3, we want "times" to be defined in such a way that (7.5 x 6.3) is more than (7.5 x L) but less than (7.5 x G). And of course the same kind of wants apply to multiplying any non-integers at all.

As you already know, the way you learned long ago to multiply 7.5 x 6.3 does satisfy the desire that (7.5 x 6.3) lie between (7.5 x G) and (7.5 x L). The kind of multiplication you know therefore does satisfy the second desire. Does it satisfy the first desire -- that it be useful?

This question can only be answered by experience, and experience shows that the kind of multiplying you know is very useful, indeed. In Experiment 3, for instance, we used the rule that the area of a rectangle can be computed by multiplying its length times its width. For that purpose, then, the kind of multiplying that you already know is useful. The fact is that that kind of multiplying is found to be useful in numberless other cases, too. This usefulness tells us that the kind of multiplying that you know is an operation worth giving a name; we call it "multiplying." We represent the operations you have to go through to multiply one number by another by the "times sign", x.

But notice that in physical situations, it often happens that two quantities pop up. Does it follow that multiplying them together will give us a useful product? Only experiment can answer that question. Physical science is founded on experiments that show us when it is meaningful to multiply two quantities together and when to do many other things with quantities.

Once we have settled on what multiplying means, dividing follows automatically. For instance, you know that (6. 3 x 7. 5) is 47. 25. But suppose you didn't know that. Suppose you wanted to know what number you have to multiply by 6. 3 to get 47. 25. You say to yourself, "Surely there must be *some* number which, when I multiply it by 6. 3, gives me 47. 25. What is that number?" We will lay aside the question -- regretfully, because it's an intriguing question -- "*Why* must there be such a number? Maybe I only wish there were one and there really is no reason to believe that it *must* exist." We will lay aside this question and *assume* that it does exist.

Now we will use symbols again. We say "There is a number that gives 47. 25 when you multiply it by 6. 3. We don't know what the number is, so we will call it Q. Then, whatever Q is, it has to be true that

$$6.3 \times Q = 47.25."$$

Furthermore, whatever Q is, it has to be found by doing something or other with the two numbers, 47. 25 and 6. 3. This "something or other" is called "division." Just as we represent the operation of multiplying by the "Times sign," we represent division by the "fraction bar", /. For instance, "6. 3 x 7. 5" means "the result when you multiply 6. 3 by 7. 5." So also, "47. 25/6. 3" means "the result when you divide 47. 25 by 6. 3." But notice one very important thing: division is *not* commutative. Although "6. 3 x 7. 5" means the same as "7. 5 x 6. 3," in division "47. 25/6. 3" does *not* mean the same as "6. 3/47. 25." Sometimes the fraction bar is written horizontally:

47. 25/6. 3 and $\frac{47.25}{6.3}$ both mean "47. 25 divided by 6. 3."

Of course you already know *how* to carry out the operation of dividing one number by another, but it is very important from now on that you know what it means. It is especially important that you know the relationship between multiplying and dividing. The relationship is very simple, but you must know it.

First, notice that 47. 25/6. 3 means the *result* when you divide 47. 25 x 6. 3. That is, 47. 25/6. 3 is *one* number, though it may look like two. Divide 47. 25 by 6. 3 to see what you get; you ought to get 7. 5. In other words, "47. 25/6. 3" and "7. 5" are merely different names for the same number. We may write

$$\frac{47.25}{6.3} = 7.5,$$

where the equals sign, as before, means "is merely another name for." Now let us take this number and multiply it by 6.3. This we can do, because we know how to multiply two numbers. Since "47.25/6.3" and "7.5" are merely different ways of writing the same number, we of course have to get the same result whether we multiply 47.25/6.3 by 6.3 or multiply 7.5 by 6.3, that is,

$$\frac{47.25}{6.3} \times 6.3 = 7.5 \times 6.3$$

Now keep in mind that $\frac{47.25}{6.3} \times 6.3$ is one number and 7.5×6.3 is also one number. Moreover, they are the same number. Multiply 7.5×6.3 and you will get 47.25. In other words, 47.25 is still another name for the number that may also be written "7.5×6.3" or "$\frac{47.25}{6.3} \times 6.3$." That is,

$$\frac{47.25}{6.3} \times 6.3 = 47.25$$

Look at this last expression. The thing on the left hand side of the equals sign means "The number you get when you divide 47.25 by 6.3 and then multiply the result by 6.3." But this result, says the thing on the right hand side of the equals sign, is 47.25. In other words, if you divide 47.25 by 6.3 and then multiply the result by 6.3, you get back unchanged the 47.25 you started with.

The result is true for any numbers at all. Let's use symbols. Suppose that A and B are any numbers at all. You know how to divide one number by another, and so you could calculate A/B if you knew what numbers A and B were. Since you don't know, we will say that you would get Q if you carried out the division. That is,

$$\frac{A}{B} = Q$$

and A/B and Q are merely different names for the same number. But remember also that "dividing A by B" means "finding that number which when multiplied by B gives A." In other words, Q is the number which when multiplied by B gives A; or $Q \times B = A$. Now we multiply both sides of the equations above by B. The products have to be equal, because we are really multiplying the same number by B. It then looks like this:

$$\frac{A}{B} \times B = Q \times B$$

Remember that we are entitled to say these two things are equal because $\frac{A}{B}$ and Q are the same number; and if we multiply that number by B, we get only one result, whether we call the result "$\frac{A}{B} \times B$" or "$Q \times B$." But $Q \times B = A$, you remember; be absolutely sure you know why!

That is, A is still another name for "$\frac{A}{B} \times B$" and "$Q \times B$." Therefore

$$\frac{A}{B} \times B = A.$$

In words: If you divide any quantity whatever by a second quantity, and then multiply the result by the same second quantity, you get the first quantity back again unchanged.

This is a very important conclusion. Notice that we proved it from the definition that A/B means a number (call it Q) such that B times Q is A. We will have much occasion to use this property that "dividing is an operation that undoes what multiplication does." Be sure you understand that this last sentence (the one in quotation marks) is true not merely because somebody says so. Notice that we proved it must be true for any numbers, because we never committed ourselves as to what numbers A and B are. Then starting with the definition of what dividing means (what does it mean?), we showed that the quoted sentence has to be true.

Let one number (say A) be divided by another (say B), to produce the quotient A/B. Then let A/B be multiplied by a third number, say C. We would write the final result.

$$\frac{A}{B} \times C.$$

Now, let A be multiplied by C to give the number A x C. Then let this result be divided by B. We would write the final result,

$$\frac{A \times C}{B}$$

You probably already know that you get the same result this time as the first time. That is, it doesn't matter whether you first multiply and then divide, or first divide and then multiply. In other words,

$$\frac{A}{B} \times C = \frac{A \times C}{B}.$$

Since "$\frac{A}{B} \times C$" and "$\frac{A \times C}{B}$" are different names for the same thing, it doesn't matter which one you write. We usually write it the second way because it seems to look nicer. But remember when you have to work out $\frac{A \times C}{B}$, it doesn't matter whether you first divide A by B and then multiply by C; or first multiply A by C and then divide by B; or first divide C by B and then multiply by A. The same idea holds even when you have a more complicated fraction like $\frac{A \times B \times C}{D \times E \times F}$. To work out this fraction you may do any of the multiplications or divisions in any order you please. But the order is important if some of the operations are addition or subtraction.

Why don't you take some complicated fraction like $\frac{3 \times 11 \times 14}{23 \times 15 \times 8}$ and multiply and divide in different orders to satisfy yourself that you do always get the same result? Carry the division to three decimal places for each trial.

6. Solving Equations

"We studied all about squares and their diagonals today," Tom told his brother Jerry when he got home from school. "We found that, for any square at all, the ratio of the diagonal's length to the edge length is always the same number."

"Oh. Well how long is the diagonal of a square?" asked Jerry.

"It depends on the size of the square," Tom replied.

"But you said the length of the diagonal was always the same for any square at all," Jerry said.

"No, I didn't. I said the ratio of the diagonal to thelength is always the same. The ratio is the square root of two. If you have a big square," Tom went on to explain, "the edge and diagonal are both big. If you have a little square, the edge and diagonal are both little. But whether you have a big square or a little one, the ratio of diagonal to edge is always the same."

"Oh, I see," said Jerry, beginning to get the idea. "If you have a bunch of different squares, you also have a bunch of different edge-lengths and diagonals. But if I take any one square and divide its diagonal by its edge, I always get the same number, no matter which square I choose."

"Right," Tom assured him.

"Then if I have two squares with different edges," Jerry said, "the diagonals have different lengths. I can see that."

"Right again. If the squares have different edges, the bigger one has to have the bigger diagonal. You see, if they had the same diagonal, then when you divided the diagonal by the bigger edge you would get a smaller ratio than when you divided the same diagonal by the smaller edge. And that's not allowed -- you must always get the same ratio. The only way you can get the same ratio is if the square with the bigger edge also has the bigger diagonal."

"Okay," said Jerry. "Then if a square has a certain edge, there is only one diagonal it can possibly have in order to make the ratio exactly $\sqrt{2}$. If the diagonal is bigger than that one thing, then the ratio would be bigger than $\sqrt{2}$ and if the diagonal were smaller than that one thing, the ratio would be less than $\sqrt{2}$. Am I right so far?"

"Yes," Tom answered, not sure what was coming next.

"A square with a certain edge can have only one certain value for its diagonal," Jerry repeated. "Then if I tell you the edge of a certain square, you should be able to tell me its diagonal, shouldn't you?" he asked.

Tom hesitated, then admitted, "I guess I ought to be able to, now that you remind me that a given square can have only some certain length for its diagonal. But I don't think I see how to do it."

Tom and Jerry were right, of course. If you fix the edge length of a square at some definite value, the diagonal length is also automatically fixed, whether you want it to be or not. You can think of all the possible squares in the world. Then when you ask "What is the edge length of a square?", immediately you realize that it can be any length at all, depending upon which square you are talking about. The edge length of a square (any square, not some particular one), in other words, is a variable. The diagonal length of a square is is also variable, because you can find a square having any length you please for its diagonal. You are free to choose, out of all possible squares, any edge length or any diagonal length you please, But you cannot do both. Once the edge length is chosen -- once you select some particular value for the edge length -- the diagonal length is fixed whether you like it or not.

Now let's see whether we can help Tom, who had a feeling he ought to be able to solve the problem his brother posed, but wasn't sure how to do it. Suppose we have a square whose edge length is 8.73 cm. How long is its diagonal? First, we admit from the beginning that we don't know (yet) how long the diagonal is; but since we want to talk about it, we'll give it a temporary name -- say D. Now when you divide D by the edge-length, 8.73 cm, you must get 1.414. (We have three significant figures in 8.73 and therefore carry $\sqrt{2}$ to four significant figures to have one extra significant figure for safety.) We therefore know that

$$\frac{D}{8.73 \text{ cm}} = 1.414$$

where "D/8.73 cm" and "1.414" are merely different names for the same number. Next, we multiply both sides of this equation by "8.73 cm." Since the two sides are the same number and we are multiplying both by the same quantity, the results must be equal. Hence

$$\frac{D}{8.73 \text{ cm}} \times 8.73 \text{ cm} = 1.414 \times 8.73 \text{ cm}$$

Look at the left-hand side of the equation -- do you see what we have done? The left-hand side says "Take D and divide it by 8.73 cm and then multiply it by 8.73 cm." By this time you know that multiplying and dividing something by the same quantity leaves that something unchanged. Hence $\frac{D}{8.73 \text{ cm}} \times 8.73 \text{ cm}$ is just another name for D. So in the equation above, we may replace the left hand side by its other name, D, and we then have

$$D = 1.414 \times 8.73 \text{ cm}.$$

But this equation tells you that the thing we didn't know, D, is equal to 1.414 x 8.73 cm, or, if you multiply it out, 12.34 cm. Now Tom knows how to calculate the diagonal of a square if he knows the edge. Do you?

If we wanted to, we could carry out the whole chain of reasoning in the preceding paragraph using symbols only. Like this: It is always true that

$$\frac{D}{E} = \sqrt{2},$$

where D and E are respectively the diagonal and edge lengths of a square. Since $\sqrt{2}$ has no units, you know from the rule for dividing quantities that D and E must be expressed in the same units -- cm, feet, miles, it doesn't matter as long as they both have the same units. We can multiply both sides of this equation by E, getting

$$\frac{D}{E} \times E = \sqrt{2} \times E.$$

The left-hand side, of course, is just D, so finally

$$D = \sqrt{2} \times E.$$

So, for any square, you can calculate the diagonal by multiplying the edge by $\sqrt{2}$. D will then be in the same units as E.

When we have an equation like $D/E = \sqrt{2}$, and manipulate it in such a way that we end up with an equation that has D all by itself on one side, we say that we have "solved the equation for D." Much of physical science deals with the solving of equations for things, like D in Tom's problem, that one feels ought to be determinable, but are buried in an unsolved equation. You will see many examples of this as we go along. Here are a couple more.

Suppose you know the diagonal of a square; can you then calculate what the edge must be? For any square, $D/E = \sqrt{2}$. This is an equation, and we would like to solve it for E; for if we had the equation in the form "E=something or other," then we could calculate the "something or other" and we would have E. Can this be done? Well, let's see.

Start with	$D/E = \sqrt{2}$
Multiply both sides by E	$\frac{D}{E} \times E = \sqrt{2}\,E$
Drop the E's on the left side because $\frac{D}{E} \times E$ is merely D	$D = \sqrt{2} \times E$
Divide both sides by $\sqrt{2}$	$\frac{D}{\sqrt{2}} = \frac{\sqrt{2} \times E}{\sqrt{2}}$

Drop the $\sqrt{2}$'s on the right side because $\frac{\sqrt{2} \times E}{\sqrt{2}}$ is merely E $\qquad \frac{D}{\sqrt{2}} = E$

Write the equation in reverse, since if X = Y, then surely Y = X. $\qquad E = \frac{D}{\sqrt{2}}$

And there you have it! To get the length of the edge of a square, you need only divide the diagonal by $\sqrt{2}$.

Do you see the idea behind this method of solving an equation? Let's try it once more, this time using symbols entirely. Suppose that A, B, and C are three quantities, and it is known that $A/B = C$. Solve the equation for A. You think like this:

1. I have $\qquad \frac{A}{B} = C$

   I want the A all by itself, so I have to get rid of the B that appears on the left. I may not just throw the B away, because I could not then be sure that the equation left would still be true. But I know that the equation would still be true if I do the same thing to both sides. Is there anything I can do to both sides that will get rid of the B on the left? Sure there is:

2. Multiply both sides by B $\qquad \frac{A}{B} \times B = C \times B$

   and now I can

3. Cross out both B's on the left $\qquad A = C \times B$

The A is now by itself and the equation is solved.

You notice we have now many times made use of the fact that an expression like $\frac{P}{Q} \times Q$ or $\frac{P \times Q}{Q}$ (where P and Q are numerical quantities) may be simplified by noticing that any time a quantitiy, P, is both multiplied and divided by the same quantity, Q, the first quantity, P, is left unchanged. This fact is often expressed by saying:: when any quantity appears, alone or as a multiplier, in both the numerator and denominator of a fraction, you may cancel out that quantity without changing the value of the fraction.

Take the equation $C/D = \pi$, and see whether you can (a) show how to compute C if D is known, and (b) compute D if C is known.

In general, you can solve an equation only if there is only one thing in it that is not known. You could not, for example, find the diagonal of a square from the equation $D/E = \sqrt{2}$ if you didn't know the edge.

Now you need some practice in solving equations. Here are some for you to work on. In all of them, remember the units.

The area of a rectangle of length L and width W is $A = L \times W$. Solve this equation for L and also solve it for W. Then find how long a strip of paper has to be if it is 2.54 cm wide and has an area of 86.2 square cm.

The area of a triangle is $A = 1/2 \times B \times H$, where B is the length of its base and H is its altitude. Solve this equation for B and also solve it for H. Also show that for any triangle, $B \times H/A$ is always the number 2. What must be the altitude of a triangle enclosing 32.7 square cm if its base is 4.96 cm long?

The area of a square is given by $A = E^2$, where E is the length of its edge. Solve this equation for E. (Here is a hint: What is the square root of E^2? That is, what must you multiply by itself in order to get E^2?) If a square has an area of 2.56 cm, what is its edge?

The volume of a rectangular parallelopiped (this is the official name for a thing shaped like a square-cornered box (like a cereal box, say)) is given by the equation $V = L \times W \times T$, where L is the length, W the width, and T the thickness. Solve this equation for T. How thick must a slab of wood be if it is 9.24 cm long and 4.14 cm wide and has a volume of 46.3 cubic cm?

If a car travels at a uniform speed, S, then the distance it can travel in time, T, is $D = T \times S$. Solve this equation for T and then compute how long it will take a body moving at a speed of 6.71 cm per second to travel 88.4 cm.

The area of a circle, $A = \pi R^2$, where R is the radius. Solve this equation for R and compute the radius of a circle whose area is 628 sq. cm. Use $\pi = 3.14$, which is correct to three figures.

The price of a pile of hamburger is given by $P = C \times W$, where C is the cost per pound (in cents) and W is the weight in pounds. What are the units of P? How many pounds of hamburger could you buy for 248 cents if the cost is 62 cents per pound?

7. Once Again, Lightly:

When one is investigating physical quantities, he soon learns that some quantities may change in value while others keep always the same value. An electric train running around a track, for instance, may speed up or slow down, so that its speed is a quantity whose value changes. Such a quantity is called a variable. The weight of the train, on the other hand, remains the same while it runs over the track, and its weight is therefore a quantity whose value does not change. Such a quantity is called a constant.

One also finds that some constants, though their values remain the same during any one investigation, may change in value from one investigation to another. The weight of the electric train, for instance, may remain unchanged in the incident mentioned above, but you know very well that the weight of a train can be changed. Such constants are often called temporary constants.

They are really variables whose values are only temporarily unchanging. On the other hand, some constants are quantities whose value never change. Such a quantity is the ratio of circumference to diameter of a circle, whose value for any circle whatever, is always π.

Just as one may do arithmetic with numbers, it is permissible to do arithmetic with quantities that are not numbers. One may add and subtract quantities only when they are measures of the same quality -- length and length, for instance, or weight and weight. But one cannot add length and weight, or temperature and time. Adding such quantities is forbidden, not by law, but by the simple fact that the sum of two unlike qualities seems to have no meaning. The sum of two quantities is another quantity of the same quality as the things added -- length plus length gives a length, for instance. The numerical part of the sum of two quantities can be obtained only if the two quantities added have the same units; the numericalpart of the sum is then identical with the number-sum of the numerical parts of the two things added.

Quantities may be multiplied or divided, however, regardless of the units they have. The product (or quotient) of two quantities is a new quantity that measures a quality which is in general different from the qualities measured by the two things multiplied or divided. The units of the product (or quotient) are therefore different from those of the things multiplied or divided. Being "new" quantities, logic says you may give their units any name you please; but convenience (so that people can talk to and understand each other, for instance) says that it is better to have rules that tell how to form the names in a uniform way.

It is important to distinguish the name of a quantity from the quantity itself. Thus a certain given stick has a certain length which is the same quantity to everybody. But one person may give this quantity the name of "36 inches", another may name it "3 feet", still another may name it "91.5 cm." These are all different names for the same quantity. When the numerical measure of a quantity is unknown or variable so that its numerical value cannot be stated, it is often convenient to give it a name not involving numbers. Such a name is called a symbol -- as the quantity representing the length of the stick above, either because it is unknown or changing, might be named "L".

When a quantity has two different names, a statement giving these two names is called an equation. The two names are called "sides" of the equation and may involve symbols and combinations of symbols as well as numbers. Using rules derived by logic from definitions, it is possible to transform an equation into other equations all of which are, by logic, known to be true. When such a transformation is carried out in such a way that a symbol which originally was buried in a combination now stands by itself as one side of the equation, the equation is then said to be solved for that symbol.

The body of rules that tell how equations may logically be manipulated belongs to mathematics. Physical science consists of finding equations to be manipulated and discovering the meanings of the new equations so obtained.

Further Classroom Discussion

Barbara measures a certain stick and later David measures it too. Pete looks at the stick and sees that it has a certain length. He asks Barbara how long it is, and she tells him 25. 5 inches. Then he asks David who tells him 67. 3 cm. Pete can see that the stick has only one length, yet Barbara and David gave him different answers. What is the trouble?

According to the rule given in Section 3 of this unit (which you also learned long ago anyway!), when you add two quantities they must be in the same units. What can it mean, then, when you are told that a certain baseball player is "six feet and two inches" tall? Doesn't this mean you are adding six feet to two inches? What about the label on the can of sauerkraut that says it contains "two pounds and three ounces"?

The most natural units of area are those formed by multiplying a length-unit by itself, like "foot-feet" or "centimeter-centimeters." (These, of course, are usually called "square feet" and "square centimeters.") The least natural are those made up out of thin air, like acre. What would the area unit, the inch-foot, be?

Similarly with volume. The cubic foot (or "foot-foot-foot") and the cubic centimeter (or "centimeter-centimeter-centimeter") are the most natural and the gallon and bushel the least. Conservationists often use the "acre-foot" to tell the volume of water in a reservoir or lake. What is an acre-foot?

Here is an English lesson for you. The unit of any quantity is a noun, and the number of the quantity is an adjective. When the number is greater than one, the unit is put in the plural. Just as you say "three men" rather than "three man", you should say "five feet" and "sixteen tons." It is improper to say "That man is six foot tall" or "I need three ton of coal." When the whole quantity (number and unit) is used as an adjective, however, the unit is put in the singular. Just as you say "There are two men on that bicycle," but "That is a two-man bicycle," so also is it proper to say "I know a five-foot quarter back" and "ten-ton truck." When the unit is a compound, you pluralize only the last member: "The reservoir holds 22 acre-feet of water."

It is also improper to use the name of the unit for the name of the quality. You do not say "What is your year?" when you mean "What is your age?"; nor do you say "This stick has a bigger foot than that one", when you mean "This stick has a bigger length than that one." What did the automobile engineer mean when he said "I bored out the cylinder in order to increase its cubic inch"?

You can determin[illegible] speed of a rifle bullet by measuring the length of time the bullet takes to travel a certain distance and dividing the distance by the time. If the distance is measured in feet and the time in seconds, what units will the speed have? Ballisticians often express the speed of a rifle bullet in "foot-seconds." How should they express it?

The speed of a ship is often expressed in knots. What is a knot?

Try this experiment. Get three bowls of water, one containing water as hot as you can comfortably stand it, one containing water as cold as you can stand it, and one at room temperature. Place one hand in the hot water and one hand in the cold water and leave them there for about 30 seconds. Then quickly remove both hands and plunge them into the water at room temperature. Make two judgements of the temperature of the water in the third bowl, one by feeling it with your right hand and one by feeling it with the left. Do they "feel" the same temperature? Is the temperature of the third bowl then a variable?

A company was founded 35 years ago, and this year three of its employes retired. Two had been with the company since it was founded and the other for 30 years. At their retirement banquet, the president of the company commented that together they represented a century of service to the company. Does it make sense to speak of a century of service to a company that is only 35 years old?

You will remember in Experiment 8 how you showed that the ratio of circumference to diameter of a circle was always the same, whether you measured it in centimeters or inches or widgets. It looked as though the ratio would be the same no matter what units you used. Of course you could not prove this to be true even by carrying out the experiment a million times using different units each time and always finding that the units made no difference in the ratio. You could never be sure that the very next unit tried wouldn't give a different ratio. (Notice, though, that you would have very well-founded reason to believe it, even though you hadn't proved it.)

Suppose that the circumference and diameter were measured in centimeters, and found respectively to be C and D. Then

$$\text{Ratio for cm measurements} = \frac{C}{D}$$

Now take some other unit of length, any at all. If this unit is a fixed amount (what good would a unit of measurement be if it were not fixed in size?) then certainly some number of them would be contained in one cm. This number might be more or less than one, but it has to be some fixed constant. Call this number N, just to give it a name, so that there are N of these units in one cm. How many of these units will there be in C cm? How many in D cm? -- if there N in one cm. The answers are respectively N x C and N x D, of course. In other words, if the circumference had been measured in the new units, since it measured C cm it would have measured N x C new units. Similarly, the diameter would have measured N x D units. The ratio, circumference/diameter, in the new units would then be

$$\text{Ratio for new units} = \frac{C \times N}{D \times N}$$

Now notice that the right-hand side of this equation is a name for the quantity $\frac{C}{D}$ multiplied by N and then divided by N. By this time you know that the result is simply $\frac{C}{D}$ unchanged. Then $\frac{C}{D}$ is another name for $\frac{C \times N}{D \times N}$ and we can write

$$\text{Ratio in new units} = \frac{C}{D}$$

where remember that C and D are the numerical measures in centimeters. We have then shown that the ratio in centimeters is the same as the ratio in any other units. Now you don't have to carry out that experiment a million times and even then not eliminate the fear that the million-and-first one may go wrong. You have proved that the ratio must be the same no matter what units you use, as long as both circumference and diameter are measured in the same units.

Unit III

Functions and Proportionality

1. What is a Function?

The idea of a function is one of the most basic ideas in all of physical science, though you must not suppose that functions are confined to physical science alone. While the notion of function is not at all a difficult one, it might be best to grow into it gradually. Basically, functions deal with relationships between two (or more) variables.

First, you should remind yourself that you often run across cases where two variables are related to each other. You are aware of such a relationship when you say that one thing depends on another. We are particularly concerned with relationships between quantities, however; that is, on one quantity's depending on another. Suppose, for instance, someone asked you how long it takes for your electric train to run around a certain track you've laid out. You would probably say "That depends ---", wouldn't you? Depends on what? Well, the time it takes the train for one circuit of the fixed track depends on how fast the train goes, doesn't it?

How long is the diagonal of a square? It depends on how long the side is. You can have any length at all for the diagonal, and any length at all for the edge; they are both variables, if you consider all possible squares. Yet once you have chosen a certain edge-length, then the diagonal length is fixed; and this is what we mean by saying that the diagonal-length depends on the edge-length. (Of course, the other way around, too.) Or, if you have a variable-speed train, you can have any length of time you want for circuiting the track (at least between certain limits; there is a fastest speed you train can travel and there may be a slowest speed too), or any speed you want. But once you have chosen the speed, the time for one circuit is fixed; you no longer have an choice. This is what you mean when you say the time depends on the speed.

Or, suppose you were walking up a ramp. How far above ground are you as you walk up the ramp? Well, it depends upon how far along the ramp you've walked. Within certain limits, you can walk any distance along the ramp you please, or you can be as high off the ground as you please. But once you have chosen a certain distance to walk along the ramp, you will be as high off the ground as that point brings you, and you have no further choice in the matter.

How far will this spring stretch if I hang a fish on it? It depends on the weight of the fish. I can have any weight fish that I please, or can stretch the spring any length that I please (within limits, of course). But if I choose a certain weight for the fish, then I no longer have any choice about the spring extension. Each weight has its own extension whether I like it or not. Also the other way around. If I choose to extend the spring by a certain amount, then there is only one weight that will extend the spring exactly that much.

Now the idea of a function is very simple. When one variable depends on another, the first one is said to be a function of the other. More exactly: when two quantities are so related that, as soon as one of them is fixed in value, the other one is too, then the second one is called a function of the first. The function is the one that is automatically fixed by making a choice for the other. In the examples above:

The time it takes for an electric train to make one circuit of a given track is a function of the train's speed.

The length of a square's diagonal is a function of its edge-length.

The distance you are above ground-level as you walk up a ramp is a function of how far along the ramp you walk.

The amount by which a spring stretches is a function of how much weight is attached to it.

It is often (but not always) true that when one quantity is a function of another, the second is also a function of the first. This is the case with each of the four discussed above.

Try to cite some other pairs of variables which are functions one of the other.

You might now go back to Section 6 of Unit II and read again the saga of Tom and Jerry. You will remember that Tom and Jerry had noticed that you can have any length at all for the edge of a square, or you can have any length at all for the diagonal of a square. But you cannot have both at the same time. Once you choose a certain value for the edge, the diagonal length is no longer subject to choice; it is fixed at some certain value whether you like it or not. The two boys then carried their thinking one step further. They said: if the diagonal of a square is fixed when I am told what its edge-length is, then I ought to be able to figure out the diagonal-length when I am given the edge-length.

What Tom and Jerry were saying is this: "The diagonal-length is a function of the edge-length. If I actually draw a square with an edge-length you give me, then the square will automatically adopt exactly the right diagonal-length. If a square is clever enough to do this without thinking, then surely I ought to be clever enough to figure it out without drawing it. Drawing it would be "cheating" because that's really the same as having the square figure it out for me."

We learned that, for all squares, the quotient, diagonal-length divided by edge-length, is always $\sqrt{2}$. This we learned by experiment. Then by logic we deduced that $D = E \times \sqrt{2}$. Thus by a combination of experiment and logical thought, we found a way of calculating the unknown diagonal-length from the given edge-length.

There you have in a nutshell the whole goal of physical science. If you know (or suspect) that some quantity is a function of some other, try to find some way to calculate the first quantity when the second one is given to you. For instance:

You drop a stone from the top of a tall building and notice that the longer you wait, the farther the stone falls (until it hits the ground). You conclude that the distance traveled by a falling stone is a function of the time of falling. Can you find out how to calculate distance fallen when time of fall is given?

You notice that the weight of a piece of copper depends upon the size of the piece; that the weight of a piece of copper is a function of its volume. Can you discover how to calculate the unknown weight of a piece having a given volume?

You observe that the distance that a spring extends is a function of the weight attached to it. How can you calculate the unknown extension of a spring to which a given weight is attached?

You can see that the distance above ground attained by a person walking up a ramp is a function of how far along the ramp he has walked. How can you calculate the height when the distance along the ramp is given?

Now would be a good time to do Experiment 10. Afterward we will have the usual

Points to Discuss in Class

How does the very nature of the experiment indicate that the height is a function of distance along the ramp? Notice that once you chose a distance there was only one height to measure corresponding to that distance.

What curious circumstance did you find about the ratios of height divided by distance for the straight ramp (last column in Table I)? Within the error of measurement, would it be fair to say that the ratio of height/distance is a constant for any one straight ramp?

What would you suggest doing on the first line where you had to divide zero by zero? What does it mean to divide something by zero? To try to see what it means, recall the definition of "dividing." The quantity, A/B, means that number (say Q) which when multiplied by B gives A. That is

$$\frac{A}{B} = Q \text{ means that } A = B \times Q$$

whatever numbers A and B might be. Now suppose that B is zero. This would mean that A/0 is the number which, when multiplied by 0 gives A. That is

$$\frac{A}{0} = Q \text{ means that } A = 0 \times Q.$$

Suppose first that A is not zero. Zero times any number is zero, isn't it? Therefore, $0 \times Q$ is zero regardless of what number Q may be. Hence A, which is equal to $0 \times Q$, must be zero. But we supposed that A is not zero. Now clearly A cannot be both zero and not zero, and something therefore must be wrong with our argument. What is wrong?

We started by supposing that A/0 is some number Q and A is not zero; and ended by finding that A is zero. Since this is not possible it must be that our supposition is wrong. Now we supposed two things:

A is not zero

A/0 is a number (which we called Q)

Certainly there is nothing wrong with making A anything we please, including 2, 17, 45.9 or any other number not zero. There is nothing wrong with the first assumption. Therefore the second assumption must be wrong: A/0 cannot be a number if A is not zero. This is worth repeating:

When A is not zero, $\frac{A}{0}$ is not a number.

This is a statement sometimes loosely quoted as "You can't divide by zero."

Now suppose that A is zero. It still must be true (using the same old definition of dividing that

$$\frac{A}{B} = Q \text{ means that } A = B \times Q.$$

If A and B are both zero, this last sentence becomes

$$\frac{0}{0} = Q \text{ means that } 0 = 0 \times Q.$$

Now notice that the last equation is true for any Q at all. Hence 0/0 may be any number at all: it is not defined or determinable. This too is worth repeating:

$\frac{0}{0}$ is not defined.

Since 0/0 is not defined by the process of dividing, you are of course free to make it mean whatever you would like it to mean under the particular circumstances where you find it occurring.

What would you like "0/0" to mean on the first line of your Table I, in order to make all the lines consistent? Now you can make the statement for all distances, including zero: "The ratio of 'height above ground' divided by 'distance along the ramp' is a constant." (When the ramp is straight!)

This question arises: Can you predict (that is, calculate without measuring) the height above ground for a given distance along ramp? You have a suspicion that it might be possible, because you ran into a similar situation before: You found that the diagonal-length of a square is a function of its edge length, and then found a way to calculate the diagonal-length when you knew the edge-length. Can you calculate height when a distance is given? Of course you can! Let H represent "height above ground" and let D represent "distance along ramp." Then from your experiment, you seem entitled to say

$$\frac{H}{D} = \text{some constant}$$

For your ramp what is this constant? Take its value as the average of the ratios in the last column of Table I. Place your value in place of "some constant" in the last equation above and then solve the equation for H. Now you can predict the height above ground that would be attained by walking along the ramp any distance you please. Calculate the height above ground for a distance along ramp of 15.00 cm. Record this value on data sheet #2 of Experiment 10, in the box "Calculated value of H for D = 15.00 cm."

Did you find the ratio for the crooked ramp in Table II also constant? May you for the crooked ramp write an expression like "H/D = some constant? Notice that for the straight ramp you may write the expression

$$H = k \times D$$

where k is some measurable constant; but for the crooked ramp you cannot. You can easily calculate H when given D for the straight ramp, but you have no way of doing that for the crooked ramp.

Now for both ramps, height is a function of distance, isn't it? Why? Because for any given distance along either ramp, there was always one and only one height to measure. In the case of the straight ramp, however, we found an easy way to calculate the function (height) from the variable (distance). For the crooked ramp we have found no way to do this. For a straight ramp, height is a known function of distance; for a crooked ramp, the height is an unknown function.

Of course, to say that the function is unknown does not mean that it is unknowable. It would be possible to write an equation, far more complicated than H = k x D, for the crooked ramp, though it hardly seems worth doing. Straight lines occur very frequently in the world, but a curve shaped exactly like your crooked ramp does not occur often enough to make it worth studying. The broad goal of physical science is to find useful relationships of this kind. That is, the goal of physical science is:

(1) To recognize what physical qualities can be measured as quantities;

(2) To seek out those cases where two (or more) variables are so related that one is a function of the other(s); and

(3) To express this function in the form of an equation.

2. Functional Relationships

When one variable is known at least to some extent as a function of another, that information may be useful. How can it be communicated? Let's visit Tom and Jerry again.

Tom said to Jerry "You know that ramp in our backyard? Well I made some measurements on it yesterday and I found that as you walk up the ramp, the height you are above the ground is a function of how far up the ramp you walk."

Jerry wasn't quite sure he knew what the word "function" meant, so he asked, "Does that mean that if I walk up the ramp a certain distance, then the height I am above ground at that point is fixed by how far up the ramp I walked?"

"Yes, that's it exactly."

"Well, I use that ramp every now and then myself, and it would be useful to me to know how far above the ground I happen to be for any distance along the ramp." Jerry got out his notebook and pencil, then said to his brother, "I want to write this down. Tell me how I can know how far above ground I am for different points on the ramp."

Tom was prepared for this, because he made a table just like your Table I in Experiment 10. He showed it to his brother, who looked at it carefully and then protested:

"But, wait a minute. This table is good for only 10 different points on the ramp. Suppose I want to know how far above ground I am when I'm standing at some distance not in your table?"

Tom was a little crestfallen. He had gone to some trouble to make the table, was proud of it, and was happy that his brother might make use of his work. Now Jerry had picked on a serious defect in it. "I could go back," he offered, "and measure another 10 points. Then you would be sure to be near one of the entries in the table no matter where you stand."

"I'm afraid that won't do," Jerry replied. "No matter how many points you measure, you can never be sure that exactly the point I need will be among them. Isn't there some way you can tell me the height for every point no matter where it is?"

Tom didn't see how he could do this right away, so let's see whether we can help him.

You already know that one variable is a function of another when they are so related that the first is automatically determined when the second is fixed. Thus if you are told that X and Y are variables and Y is a function of X, then you immediately know that Y has some fixed definite value when you assign at your pleasure some definite value to X. The variable (X in this case) to which you assign any value at pleasure is often called the independent variable -- "independent" because you may assign its value independently of anything else. The function (Y in this case) is often called the dependent variable -- "dependent" because its value depends on the already-assigned value of the other and cannot be assigned at pleasure. Of course, as we noted before, when Y is a function of X, it is often true that X is a function of Y. In such a case, it may be convenient to call Y the independent variable and X the dependent one. The two terms, independent and dependent variable, are used only as a matter of convenience. In much the same way, you can imagine a conversation like this:

"Did you know that Smith has a brother?"

"Yes. Smith is a good friend of mine but I hardly know his brother."

The participants in this conversation have quite clearly in mind which person is Smith and which is his brother; yet the truth is that both these persons are Smith and both are Smith's brother.

Then if Y is a function of X, we can assign at pleasure a value to the independent variable X; and know that the dependent variable Y automatically has its value fixed. But to know that Y has some definite value when X is fixed at, say, 10.07 cm, is a far cry from knowing what that value is. In other words there is a difference between knowing merely that Y is a function of X, and knowing exactly what function of X the dependint variable Y is. Any means of telling what function of X some other variable Y is, is called a functional relationship.

One way of communicating a functional relationship between two variables is by a table. Such a table would give a selected list of values for the variable X; and opposite these selected values of X would be listed the corresponding values of the function Y. Such a tabular representation of a function, however, has the very serious defect that Jerry had put his finger on in the exciting drama above. Even if the table stretched in fine print from here to the moon, it could not give the value of Y for every possible X. By its very nature, a tabular representation of a functional relationship can give only a limited number of values of the two variables. This is quite satisfactory in some cases, especially in cases where the independent variable is, by its very nature, one that can have only a certain number of values. An example might be

Independent variable: A year of the twentieth century

Dependent variable: The total rainfall that year in Dallas.

Here a table would be complete, because there is no year 1958.5. Such a variable is called discrete: no two possible values of the variable can be any closer than a certain amount (in this case one year) apart. Our concern here is with continuous variables, those in which you can have two values of the variable as close together as you please. The points on a line, for instance, may be 1 cm apart, or 0.001 cm, or 0.0000000001 cm, er even closer than that if you wish.

3. Graphs

One can get around the discrete nature of a tabular representation by using a graph. As you already know, you make a graph by plotting a series of points with respect to two axes. You choose a certain length on the horizontal axis to represent one unit of the independent variable and a certain length along the vertical axis to represent one unit of the dependent variable. You then take one pair of values of the two variables and locate the point on the graph in much the same way as you locate a city address as the intersection of two streets. When all the points have been plotted, you connect them by a suitable curve.

Now is the time to finish Experiment 10.

Points to Discuss in Class

When one constructs a graph from a set of points plotted from experimental data, the problem always arises: How shall I draw a line through these points to complete the graph? You have this problem right now in the graphs from Experiment 10. (In fact, you were told to draw the lines lightly in pencil because they are only temporary: we want to discuss what you should do before doing it permanently.) The "line" drawn through experimental points of a graph is called a curve, and this is true even when the "curve" is a straight line!

Before you can sensibly decide how to draw the curve, you must first understand why you draw the curve at all. Why do you? Keep in mind that a curve is a functional representation: it is supposed to tell you the value of the function, H, for chosen values of the independent variable, D. But the plotted points alone give no information not already in the table; they only present that information in a different way. Why draw the graph?

One reason for drawing a graph is that you can view the whole set of points at once and comprehend their relationship more easily in a picture than you can in a table. [In the same vein, you realize that a map of your state that has all the cities and towns "plotted" on it is easier to comprehend than a table that lists the latitude and longitude of every city and town -- yet both map and table give exactly the same information.]

Another reason for drawing a graph is to satisfy Tom and Jerry's problem: How can I find out the value of H (the dependent variable) for values of D (the independent variable) that I didn't measure? In other words, our measurements of H and D must necessarily be limited in number. We therefore make D a discrete variable -- by spotting in only a few chosen values -- when it really is a continuous variable which may have any value at all. It may have all possible values between any two of the values you happened to choose to measure. One of the purposes of making a graph, then, is to supply the in-between values which you could never fill in completely if you and all your classmates worked from now until Doomsday without taking out time to eat, sleep, or play pinochle.

Now, how can you supply the in-between values if you don't measure them? The strictly logical answer to this question, of course, is "You can't"! But that answer is so disappointing that we look elsewhere for help. Think again about that ramp -- either the crooked one or the straight one. Imagine yourself a little man walking along the ramp. You find that 5 cm from the bottom places you 1. 5 cm from the ground and 6 cm from the bottom places you 1. 8 cm from the ground. You now have a feeling that if you stood 5. 5 cm from the end you ought to find yourself somewhere between 1. 5 and 1. 8 cm from the ground. Of course, it is entirely possible that the ramp could make a sudden dip between 5 and 6 cm from the bottom so that at 5. 5 cm you would perhaps even be flat on the ground. But even in this case you have a feeling that if you took measurements sufficiently close together -- yet not infinitely many -- such irregularities would eventually reveal themselves, and you could obtain the true tendency by assuming that the true value of the function at an intermediate point lies between its value at two nearby surrounding points. Faith in this principle stands very importantly as a foundation of physical science. It even has a name; the principle is often called the "principle of continuity." One way of stating it is to say "In the absence of reason to believe otherwise, a small change in an independent physical variable will produce only a small change in a variable dependent upon it." The principle seems so reasonable that all physical scientists place almost unquestioning faith in it. [It is only fair to say, however, that occasionally important physical happenings are over looked because an experimenter places too much faith in the principle of continuity.] Without using the principle, we could never predict anything. Noone would ever attempt to build a bridge, for instance, because the engineer would always say: "No, I won't be responsible for building this bridge. Noone ever built it before. Therefore I don't know that it will be safe, and it would cost too much to build it only to find out whether it is safe."

When you draw a graph, then, you suppose that intermediate points will lie between their nearby points on each side. The word "between" is not exactly defined (indeed the whole principle of continuity is not exact), but has the general meaning indicated in the discussion above.

One way to satisfy the principle might then be to connect successive plotted points by straight lines. Do you see anything wrong with doing so? Suppose you did so and then decided to measure another experimental point between two already taken. If this point did not fall exactly on the straight line you drew, then you would have to draw two new straight lines in place of the one you had. In other words the character of the curve would change if you took one additional measurement. Does it make sense that the character of a functional relationship of Nature would depend upon whether you made 10 or 11 measurements? Isn't it more sensible to look at the points and decide that the shape of the curve you should draw is already partly outlined by the way the points seem to form their own curve? Of course the "curve" they seem to form may be a straight line. The first rule then is draw in the graph in such a way as to follow the curve that the points themselves seem to outline.

The points you plot are laid down from experimentally measured quantities. Is an experimental quantity ever known exactly? Is there a possibility then that the "true" curve would not go exactly through some of your points? Why not? The second rule in drawing a graph then is to be guided by the plotted points, but if one or more of them appear to be "off" from the curve that the rest seem to outline, do not be afraid to draw the curve so that it misses the off-points.

Do you notice anything special about the plotted points for the straight ramp in your graph for Experiment 10? You know that the curve must go through the origin (where H and D are both zero) because if one does not go up the ramp at all (D = 0), he experiences no rise off the ground (H = 0). The rest of the points should fall on a straight line passing through the origin. Erase your lightly penciled line for the straight ramp and use a ruler to draw one straight line that connects all the points as best you can. Remember that you must expect some (or even most) of the points to be not quite on the line. Try to draw the line so it goes "down the middle," leaving about as many plotted points on one side as on the other.

Do the points for the crooked ramp seem, even allowing for a reasonable amount of experimental error, to form a straight line? They shouldn't. Erase the lightly penciled curve for the crooked ramp, and using the suggestions above for drawing curves for graphs, sketch freehand a curve through these points. It might be a good idea to use a different color pencil for the second curve.

Finally, read from your graph for the straight ramp what H should be when D = 15.00 cm. Record this value in the box marked "Graphical value of H for D= 15.00 cm" and compare it with the calculated value in the box above it.

4. Monotonic Functions

You often state a dependence between two variables in sentences like the following:

> The farther I walk along the ramp, the higher I get above the ground.
>
> The larger the volume of a piece of copper, the greater is its weight.
>
> The heavier the fish that I hang on this spring, the more the spring extends.
>
> The longer the piece of wire I cut from this spool, the more it weighs.
>
> The longer the edge of a square, the longer is its diagonal.
>
> The greater the diameter of a steel ball-bearing, the greater is its weight.
>
> The longer the time a rock has been falling, the farther it has fallen.

Do you see how all these statements are similar?

Each of the statements cites two variables, first an independent variable and then a dependent variable which is a function of the first. To make sure you see this point, it would be a good idea for you to restate each of these statements in the form: Dependent variable is a function of independent variable. For example, the first sentence would read "My height above the ground is a function of how far I walk up the ramp." Now you restate the others.

But each of these statements says more than simply that one of the variables is a function of the other. It tells partly the nature of the function. The first one says not only that height is a function of distance, but also that as the distance along the ramp increases, so does the height above the ground. Each of the statements says that the two variables are so related that increasing the value of the independent variable causes the dependent variable to increase, too. Stated in this form, the third sentence above says "Increasing the weight of the fish hanging on this spring causes the extension of the spring to increase." Try your hand at recasting the other sentences in this form.

There would be no point in calling all this to your attention if it were true that all functions behave like this; that is, if all functions were such that increasing the independent variable causes the dependent variable to increase, too. But all functions do not behave like this. For instance:

The farther I walk down this ramp, the less is my height off the ground.

The harder I squeeze on this spring, the shorter it gets.

The greater the speed of my electric train, the less time it takes to circuit the track.

The greater the diameter of a round cake pan, the less the height to which a pint of batter will rise when poured into it.

The greater the diameter of a wheel, the fewer revolutions it will make when it rolls a hundred feet.

See how well you can recast these sentences in the form "The height I am above the ground is a function of how far I've walked down the ramp."

Notice again, however, that each of these new sentences says more than merely that one variable is a function of another. Each also tells something of the nature of the function. For instance, the secondone says "As the squeezing force exerted on this spring increases, the length of the spring decreases." You shouldnow recast each of the other sentences in this form, just to be sure that you are getting the point.

So you see that the word function is not a very explicit word. To say that Y is a function of X is to say only that Y has a definite fixed value when the value of X is fixed. The implication in general is that when X changes, Y is forced also to change. But to say merely that Y is a function of X does not say anything about how Y changes when X changes. In some functions, when the independent variable increases, so does the function (or dependent variable). In other functions, when the independent variable increases, the function decreases

A function of the former kind (one that increases as the independent variable increases) is called an increasing function of its independent variable. The first example on page 65, for instance, says "My distance above the ground is an increasing function of the distance I walk along the ramp." Now you recast each of the other examples in the first group in this form.

In contrast, a function of the kind in the second group of examples (page 66) -- one that decreases as the independent variable increases -- is called a decreasing function. The second example in this group, for instance, says "The length of this spring is a decreasing function of the squeezing force applied to it." You should now rephrase the other sentences in this group in such a way as to use thephrase, "decreasing function."

A function that is either an increasing function or a decreasing function is called a monotonic function. Can you see where the word "monotonic" comes from? When two variables are so related that, as the independent variable increases, the function always increases or always decreases, then the latter is called a monotonic function of the independent variable. The idea of a monotonic function is very simple: it is one that always changes in the same direction as you increase the independent variable. The function may either increase steadily or decrease steadily, but it never changes its direction.

Now,you might be saying to youself "Why all the fuss about calling a function monotonic? I can see that when you increase the independent variable, the dependent variable either increases or it decreases. Why bother to drag in the adjective 'monotonic'? A function has to be either increasing or decreasing, doesn't it? Then any function must be monotonic, so why use this unnecessary word?" There is a reasnn; do you see it?

The truth is that there are many functions that are not monotonic. Of course, no variable can both increase and decrease at the same time; but it is entirely possible that a function may at first increase as the independent variable increases, and then later decrease. Or vice versa. For instance:

Think of an arched ramp, and how your height above the ground changes as you walk continuously in one direction along the ramp.

A baseball thrown directly upward will at first increase its height as time goes on, then it will reach its highest point, then decrease height as it falls back to earth.

Think of the tip of the minute-hand on a clock at exactly one o'clock. As time goes on the tip of the minute-hand descends until it reaches its lowest point at one-thirty but then it begins to ascend again.

Think of an empty drinking glass, open end at the top, and push it slowly downward in a pail of water. What happens to the water level in the pail as you push the drinking glass slowly downward?

These are functions that are not monotonic; can you think of still others? (Of course, any function has to be monotonic in a little limited portion of the range of the independent variable.)

Now go back to Experiment 10 and look first at the two data tables. Each of these tables, you will recall, gives H (height above the ground) as a function of D (distance from the bottom of the ramp). Do your two tables show that H was an increasing function of D? Now look at the graphs you made of these two functions. What do you see as a characteristic of a graph of an increasing function? How would the graph have looked if the ramp had sloped downward rather than upward? How would the graph have looked if you had used an arched ramp? The graph of an increasing function always slopes . up to the right. The graph of a decreasing function always slopes down to the right. The graph of a monotonic function always slopes in one direction -- either generally up or generally down to the right. The graph of a function that is not monotonic has what immediately seen characteristic? This is one of the main uses of a graph: it allows you to see clearly and immediately the general behavior of the function, which may be quite a chore to dig out from the tabular representation.

5. Proportionality

The two graphs from Experiment 10 bothrepresent increasing functions -- they both slope upward everywhere to the right. But you notice that they differ in one important respect: the curve for the crooked ramp is somewhat curvy whereas the curve for the straight ramp is straight. The curve for the crooked ramp has a changing slope; sometimes it slopes only very gently upward -- almost flat -- and at other times it becomes more steep. The slope is always upward to the right, to be sure, but is sometimes more and sometimes less steep. The curve for the straight ramp, in contrast, never changes its its slope; it never becomes more nor less steep, but keeps on going with the same steepness everywhere. You recognize then that we can think of two classes of increasing functions: that simple kind in which the increase is steady with a never-changing slope; and a more complicate kind in which the slope is always increasing but yet changes so as to become sometimes more and sometimes less steep. Do you see how the graphical representation of these two functions reveals this character so much more easily than the tabular representation?

But now go back to the two data tables in Experiment 10 and look at the ratios of H/D calculated there. Recall that for the straight ramp we found that H/D is a constant whereas for the crooked ramp H/D was not constant. Again we can think of two classes of increasing functions: a simple class in which the ratio of "dependent variable divided by independent variable" is constant, and a class in which this ratio is not constant. Both are increasing functions, you understand. But in the simple class the two variables, dependent and independent, are so locked together that, no matter how they change, they always do so in such a way that their ratio remains unchanged. In the other class, the two variables are also locked together so that a certain assigned value of the independent variable fixes unarguably the dependent variable, but their ratio does not remain constant.

Observe that we have classified increasing functions by two different schemes. In one scheme, we lumped into one class those that had straight graphs and into the other those whose graphs are not straight. In the other scheme, we placed into one class those functions in which the ratio of "function/independent variable" is constant, and into the other class those for which this ratio is not constant:

	Scheme #1	Scheme #2
One class	Graph is a straight line	Graph not a straight line
Other class	Ratio of dependent/independent is constant	Ratio of dependent/independent not a constant

Notice too that for both ramps you found Height to be a function of Distance. But for the straight ramp you found this function to belong to the first class according to both schemes of classification and the crooked ramp to belong to the second class in both schemes. The intriguing question comes up: is this always true? That is, is a function whose graph is a straight line always a function whose ratio of dependent/independent is a constant; and is a function whose graph is not a straight line always one for which this ratio is not constant?

Now here is a very important bit of logic. Simply because we have set up two different schemes of splitting a set into two halves, it does not follow that the two splits are identical. There are lots of ways of cutting an orange in half.

For instance, you might split your class into two groups: boys and girls. Someone else may split them into two groups in a different way: say those who have had measles and those who haven't. You might then find a certain boy who did have measles and a certain girl who did not, and leap to the conclusion that the two splits are identical: the boys are the ones who had measles and the girls the ones who did not. Of course, you know that you cannot jump to that false conclusion.

Similarly here. The fact that we have two schemes for classifying functions, and have found two cases where the splitting is identical does not mean that it is always so. But it is still an intriguing question and we ought to look into it further. That's one purpose of Experiment 11, which you ought now to do.

Points to Discuss in Class

Suppose you wanted to measure the position on a vertical yardstick of a pencil that you are holding two inches in front of the yardstick. If you held your eye at exactly the same horizontal height as the pencil, you would get one reading. If you held youreye above the level of the pencil you would get another reading; and if you held your eye below the level of the pencil, still another.

Which is correct? Why was it important in the experiment to sight horizontally across the hook to the ruler? The error made in a measurement because the eye is not in the correct position is called an error of parallax. You can see how important it would be in careful measurements to avoid this error. It must always be considered when, as in this experiment, the ruler and the thing to be measured cannot be brought together.

What did your calculations show about the ratio, extension divided by load, for the springs you used? Did everyone in your class find this ratio constant? What are the units of this ratio? Would it be correct to summarize the results of this experiment as found by your whole class in this statement: "The ratio of the extension of a spring to the weight producing that extension is a constant"? In fact, this statement has been found by numerous experiments to be true for all elastic bodies. This finding is often called Hooke's law, after Robert Hooke who first stated it as a general rule of Nature in about 1660.

If we represent the extension by E and the attached weight by W, Hooke's law can be written more succinctly as

$$\frac{E}{W} = k,$$

where k is some constant. Show that a completely equivalent way of saying the same thing is to write

$$E = k \times W.$$

This last equation is worth special attention. Notice that it is solved for the dependent variable. This equation then gives two different names for the dependent variable: one, of course, is E itself; the other is $k \times W$. Thus the expression "$E = k \times W$" is still another functional representation of E, telling exactly what function of the independent variable W, the dependent variable E is. You now know of three types of functional representations: tabular, graphical, and this last one in the form of an equation. When a function is represented by an equation solved for the dependent variable, it is called an analytical representation. Try to discuss various advantages and disadvantages of graphical, tabular, and analytical representations of functions.

You found experimentally that E/W for spring is a constant. When $W = 0$, E of course is also zero, because if there is no load there is no extension. What will you do about the ratio 0/0 for the first line of each table in Experiment 11?

What kind of constant is the k in the expression $E = k \times W$? Is k a constant for any one spring? Do all springs have the same constant, k? Remember that k is simply another name for the ratio, E/W. Suppose that I attach a certain load, say 10 grams, to two different springs, one a weak one and the other a stiff one. Which spring will give the greater extension? Which spring will have the larger value of k? Suppose I have two springs, a stiff one and a weak one, and find they are both extended the same amount, say 10 cm. Which is supporting the greater weight?

Look at the expression, "$E/W = k$". Suppose I have two springs, one weak and one stiff. On each I hang the same weight, W. Will the extensions be the same for the two springs? Call one extension E_w (meaning extension of the weak spring) and the other E_s (meaning extension of the stiff spring). For each spring, E/W must be some fixed value, k, which is not the same for different springs. Call them k_w for the weak spring and k_s for the stiff one. We now have

$$E_s/W = k_s \qquad \text{and} \qquad E_w/W = k_w$$

for stiff spring   for weak spring

If W is the same for each spring and E_s is less than E_w (because a stiff spring extends less than a weak one under the same load), then how will k_s and k_w compare? Make a general statement: "Of two springs, the stiffer one will have the ___________ value of its spring-constant, k."

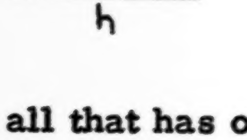
Or, suppose you have two springs, one stiff and one weak, and extend them both the same amount, by hanging different weights on them. Which will need the greater weight? Let us call W_s and W_w the two weights that will extend the springs the same amount, E. Then

$$E/W_s = k_s \qquad \text{and} \qquad E/W_w = k_w$$

If E is the same for each spring and W_s is greater than W_w (because a stiff spring needs more weight to extend it the same amount as a weak spring), then which will be larger, k_s or k_w? How does this compare with the general statement you made at the end of the last paragraph above?

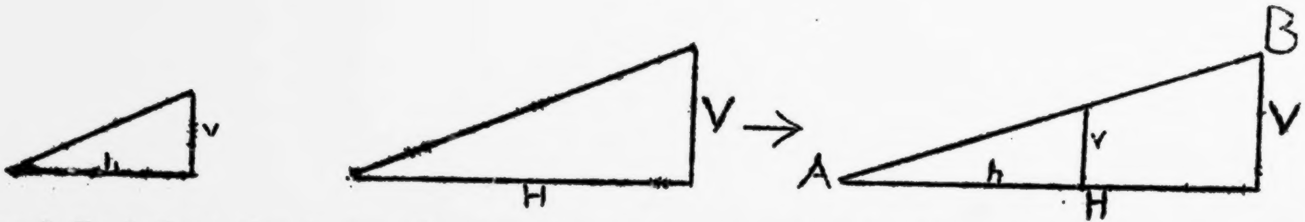
Now look at the graphs you made for the two springs in Experiment 11. Did you get essentially straight lines in both cases? Here then is another case where a function whose ratio to its independent variable is constant gives a straight-line graph. You now have a little more reason than before to believe that a function will have a straight-line graph if the ratio of function to dependent variable is a constant. Of course we still haven't proved it (why not?), so let's look into the matter a little more carefully.

Suppose you have a triangle with one right angle and arranged like this:



This may be any triangle at all that has one right angle. Let us call the horizontal leg of the triangle h and the vertical leg, v. If we measure h and v in cm, we will of course get some one certain number for the ratio, v/h. Suppose we now magnify this triangle exactly 3.694 times. The triangle will of course still be the same shape, but 3.694 times as big.

The horizontal leg will now measure (3.694 x h) cm and the vertical leg will measure (3.694 x v) cm. The ratio (3.694 x v)/(3.694 x h) is still the same because the 3.694 will cancel out of numerator and denominator. Now you can see that the same would be true if we had used 2.241, or 89.643, or π or $\sqrt{2}$, or any other number for the scale of magnification; for whatever the scale is, the number will always appear in both numerator and denominator and therefore cancel out. In other words, magnifying a triangle will not change its shape, and will keep the ratio of vertical leg to horizontal leg unchanged. Now if the triangle does not change in shape when it is magnified, then we can fit the unmagnified and magnified triangles together like this,



and find that the long sloping line is one straight line, AB. This will be true whatever the shape of the initial triangle and whatever the scale of magnification. You then see that if AB is a straight line, the ratio v/h equals the ratio V/H and this will be true for any straight line AB whatever and no matter what the positions of the two lines v and V might be. The converse is also true: if the ratio v/h equals the ratio V/H, then AB is a straight line. Do you see that, if the last two statements are true, then we can say:?

> A function whose graph is a straight line through the origin is a function whose ratio to its independent variable is constant; and
>
> A function whose ratio to its independent variable is constant always has a graph that is a straight line through the origin.

(We have discussed in the long paragraph above the matter of equivalence between a straight-line graph (through the origin) and the constancy of ratio between a function and its independent variable. This discussion is not a proof, though it is nearly one. Perhaps you can see some of the faults that keep it from being a proof. The main one is our supposition that "magnifying" a triangle by making two of its legs a certain number of times bigger would leave the "shape" unchanged. When you study geometry you will learn how to prove this without faults. Our discussion only tries to make it seem reasonable. Also, though we "nearly proved" the statement "If AB is a straight line, then a certain ratio is constant." We didn't even attempt to prove "If the ratio is constant, then AB is a straight line." Proving one does not prove the other. (Does "Every ginkle is a foop." mean "Every foop is a ginkle"?) It happens, however, that both the indented statements just preceding the present paragraph are true. You will actually prove them when you study geometry.)

When two variables are so related that their ratio is a constant, the variables are said to be proportional to each other. If Y is a function of X such that Y/X is any constant, then Y is said to be proportional to X, or X to Y. Notice that there is nothing to prove here. The following two statements say the same thing because of the phrase "is proportional to" is defined that way:

"A is proportional to B" __means__ "A/B is a constant."

Using the expression "is proportional to," restate Hooke's law for a spring. Also, make similar statements for the diagonal and edge of a square, and the circumference and diameter of a circle.

You should try to get a comfortable and clear feeling for the meaning of proportionality. Suppose that A and B are two variables such that A is a when the value of B is b. Then a/b is some constant, say k. Now suppose we double the value of B so that its value is now 2b. The value of A must now be such that when it is divided by 2b we still get the same constant k. What must be the value of A so that ?/2b = k? We know that a/b = k, so that we can write

$$\frac{?}{2b} = \frac{a}{b},$$

where "?" stands for the value that A has when B has the value 2b. Now remember that ?/2b is just a number and a/b is just a number, and the equation says they are the same number. Multiply this number by 2b; we must get the same result when we multiply ?/2b by 2b as we get when we multiply a/b by 2b, simply because ?/2b and a/b are really the same number. Then

$$\frac{2b \times ?}{2b} = \frac{2b \times a}{b}$$

Now you know that you may cancel out of numerator and denominator anything that appears in both. Then cancel 2b from numerator and denominator on the left and b from numerator and denominator on the right. You have

$$? = 2a$$

In other words, the value that A must have when B has the value of 2b is 2a. That is, doubling B requires that A be doubled. You can see that the same thing will happen if you triple B, halve it, or multiply it by 4, 0.52, or any other number.

Thus an easy way of looking at a proportionality is this: Two variables A and B are proportional when their behavior is such that multiplying one of them by some number automatically causes the other to be multiplied by the same number.

6. __The Proportionality Constant__

When one variable is proportional to another, their ratio is constant. This constant is of course dependent upon what two variables you are considering; to say that the ratio of two proportional variables is constant does not mean that this ratio is the same regardless of what variables you are talking about. The constant that is the ratio of diagonal-length to edge-length for a square is not the same constant as the ratio of circumference to diameter for a circle. Even when you deal with Hooke's law, the constant is not the same for one spring as for another. But as long as you are talking about one particular shape, the squre, the ratio of diagonal to edge is always the same; as long as you are talking about one particular shape, the circle, the ratio of circumference to diameter is always the same; as long as you are talking about one particular spring, the ratio of extension to weight is always the same.

We have seen that the statement "Y is proportional to X" means that, no matter how Y and X may change, both the following statements must be true:

$$Y/X = k \qquad \text{and} \qquad Y = k \times X$$

where k is some constant whose value depends on what the variables Y and X happen to be. Notice carefully that the two equations above are completely equivalent. Neither equation carries any information not contained in the other. This must be true because either may be derived from the other (Can you still carry out this derivation?) purely by logic without bringing in any new information. If no new information is brought in when deriving the second equation from the first, then clearly the second equation cannot contain any information not contained in the first.

You see then that saying "Y is proportional to X" not only says that their ratio is constant; it also says that I can obtain Y when you tell me X merely by multiplying the X you give me by the constant. And I can do this for any X you give me using always the same constant. This constant is called the constant of proportionality (or proportionality constant). In other words, when one quantity is proportional to another, their ratio is called the constant of proportionality.

Now suppose I tell you: "Y and X are two variables that are proportional to each other." You immediately infer that their ratio is a constant, don't you? But then you think a little and realize that the quantities, Y and X, have two ratios. One of them is X/Y and the other is Y/X. Which of these two ratios is constant? When one says that Y and X are proportional, which of the two possible ratios is the one that is constant? The comfortable answer is "Both are constant." In other words, when someone tells you that X and Y are proportional, you don't have to worry whether he means that Y/X is constant or that X/Y is constant. If one is constant, the other must be.

Do you see why? The reason is quite simple. Suppose that

$$\frac{X}{Y} = k$$

where X and Y are any two numbers whose ratio is k. If this equation is true, you have already shown that

$$X = k \times Y$$

where "X" and "k x Y" are merely different names for the same quantity.

Let us divide this quantity by k; we get the same result whether we divide X by k or (k x Y) by k, because X and k x Y are really the same quantity. Therefore the results must be the same quantity under different names, or

$$\frac{X}{k} = \frac{k \times Y}{k}$$

But here's the old story again: we can cancel out the two k's on the right hand side and write

$$\frac{X}{k} = Y,$$

where X/k and Y are different names for the same quantity. Divide both sides of this equation by X. Then you have

$$\frac{X}{k \times X} = \frac{Y}{X}.$$

Now we could cancel out the X's on the left, but a new trouble arises. Doing so will leave us with a fraction on the left that has no numerator, and therefore has no meaning. There is an easy way around this trouble; we just have to be sure that something will be left in the numerator after the X's are canceled out. We can be sure of this by putting something in the numerator. But clearly we cannot put any old thing in there. We must insert something that will leave its value unchanged, so that even after it is inserted, it will still be equal to the right hand side. Do you see that we can put a 1 there? For doing so merely means multiplying by 1, and any number may be multiplied by 1 without changing its value. Then we have

$$\frac{X \times 1}{k \times X} = \frac{Y}{X};$$

or canceling out the X's, we have

$$\frac{1}{k} = \frac{Y}{X}.$$

Thus we have shown that if X/Y = k, a certain fixed number, then Y/X = 1/k. But if k is a fixed number, then there is only one result you could get by dividing k into 1. Hence 1/k is a constant if k is. Therefore if X/Y is a constant, so is Y/X, and it doesn't matter which ratio you take as constant when someone tells you that X and Y are proportional.

But you have to be careful. Suppose someone tells you "X and Y are proportional to each other and their proportionality constant is 7.17." Does he mean that X/Y is always 7.17 or that Y/X is always 7.17? You cannot tell. Both ratios are constant, fo course, but you don't know which one is 7.17. Thus when you tell someone the proportionality constant between two proportional variables, you must always tell him which way the division is to be carried out. Thus if I say "The circumference and diameter of a circle are proportional to each other and their proportionality constant is 3.14", you might wonder what I mean. In this case a little thought would tell you that I must mean "circumference divided by diameter" because you know that the circumference is always larger than the diameter and dividing diameter by circumference would give something less than 1 and could not therefore give 3.14. On the other hand, I could say to you "The extension of this spring and the weight you hang on it are proportional to each other and the proportionality constant is 7.17 cm/g." Because the units are given, there is no question but that I mean the ratio "extension divided by weight," because the only way you can get units of "cm/g" is to divide centimeters by grams. But the direction of dividing must be given, either explicitly or by implication.

The proportionality constant in the functional relationship between two proportional variables is often itself of interesting physical significance. Let us look at a few cases in order to acquire a feeling for the meaning of proportionality constants. In each of the three illustrations following, you should first read the introductory quotation and make sure you see clearly what it means. Restate it in other words; tell yourself definitely what the two variables are; tell yourself which is the dependent and which the independent variable; try to see clearly the sense and significance of a statement like "If I double the independent variable, the dependent variable will automatically double"; and try to see whether this latter statement is in accord with your common sense and experience. Repeat: do this for the introductory quotation in each of the following examples.

1. "The price of a pile of hamburger is proportional to the weight of the pile." If we let P be the price (in cents) and W be the weight in pounds, then P/W is a constant. What is the meaning of this constant? Suppose that you pay 120 cents for a pile weighing 2.5 pounds. Then

$$\frac{P}{W} = \frac{120\text{ cents}}{2.5\text{ pounds}} = 48\text{ cents/pound}$$

In other words, the proportionality constant in this case is simply the price per pound, which is the same for any amount you buy (not considering quantity discounts).

2. "The distance traveled by a uniformly moving car is proportional to the time it travels." Let us call the distance traveled, D, and the time of the trip, T. Then D/T is a constant. If, for instance, the car travels 105 miles in 3.5 hours, we have

$$\frac{D}{T} = \frac{105\text{ miles}}{3.5\text{ hours}} = 30\text{ miles/hour.}$$

Here the proportionality constant is simply the speed of the car.

In each of these two cases, the proportionality constant has a familiar meaning. In the ratio P/W, you have simply the unit cost; in your ordinary everyday thinking, the higher the constant P/W, the more "expensive" is the material concerned. In the ratio D/T, you recognize the proportionality constant as the speed. In your ordinary thinking, the greater is the constant D/T, the faster the car has been moving. In the next case, the ratio may be less familiar but the thinking is exactly the same.

3. "The extension of a spring is proportional to the weight attached to it." If a spring stretches E cm when a weight of W grams is attached to it, then E/W is a constant. If in a particular case the spring stretches 8.76 cm when a weight of 5.34 g is attached, then

$$\frac{E}{W} = \frac{8.76 \text{ cm}}{5.34 \text{ g}} = 1.64 \text{ cm/g}.$$

Here the proportionality constant may be a little less familiar but try to see its resemblance to speed and unit cost. Here the units of the proportionality constant are "centimeters per gram." Just as unit cost means the price you must pay per pound of hamburger; just as speed means the distance you travel per hour of driving; so does the proportionality constant here means the distance the spring stretches per gram of weight hung upon it. The larger the value of this constant, the more the spring stretches per gram attached, or the more stretchable it is. Here the proportionality constant conveys an idea of the stretchability of the spring, in much the same way as unit-cost and speed are measures of costliness and speediness.

Notice a very important point: the idea of "stretchability" may have been a vague notion in your mind, hardly at any rate a numerical one. If someone asks you "How long is this stick?" or "How heavy is this rock?", you immediately bring to mind numerical answers and might reply "Six feet" or "Five pounds." But if someone had said "How stretchable is this spring?", you probably would have had no thought of numerical measure and might have answered "Oh, rather limber" or "Pretty stiff." But if you are building a machine that requires a spring of just the right stiffness, you don't send an order to a spring-manufacturer for "One spring of just the right stiffness" and expect to get what you need. You must somehow designate numerically how stretchable the spring is to be. The proportionality constant in Hooke's law is a nemerical measure of stretchability. Now if someone asks you "How stretchable is this spring?", you need not be vague; you can give him a numerical answer "17.2 centimeters per gram."

Now one final point in connection with proportionality constants. Recall again that "A is proportional to B" means

$$A = k \times B$$

where k is some constant regardless of what A and B might be. If someone should ask you, "All right, A is proportional to B, and the proportionality constant A/B is k. I want to know what value A would have when B has the value b. What is it?" Since A is a function of B, the question must have an answer because giving a value to the independent variable must give a definite value to the dependent variable. (Why is this true?) The answer of course is given by the last equation above. To find what value A has when B is b, you just multiply k x b. Suppose the question is asked the other way around: "What value must B have in order to give A the value a?" This question is also easy to answer. You should now be able to solve the above equation for B and show that

$$B = A/k.$$

Be sure that you can show this.

For instance, suppose you have a spring whose constant is 1.64 cm/g. Suppose the extension of the spring is 5.86 cm; what weight must be hanging on the spring to produce this extension? From the given spring constant, you know that $E/W = 1.64$, when E is in centimeters and W in grams. Then

$$W = E/1.64$$

and all you need do to find the weight is to divide the extension by 1.64. In this way the spring becomes a weighing machine.

Now you are ready to do Experiment 12.

Points to Discuss in Class

Within the error to be expected in your measurements, did you find the ratio, weight/length, constant for one size of rod? Did both sizes of rod give the same proportionality constant? May you reasonably conclude that, for an aluminum rod of given cross-section, the weight of the rod is proportional to its length?

What is the meaning of the proportionality constant, weight/length? Think back to the measurements you made (Experiments 6 and 8) on squares and circles. You found that the ratio of diagonal/edge for a square is the same for all squares regardless of size. The value of this ratio is purely a property of being square, and does not depend on how big the square is. You found that the ratio of circumference/diameter for a circle is the same for all circles. The value of the ratio is purely a property of being a circle, and does not depend on how big the circle is. Experiment 12 showed you that the ratio of weight/length for an aluminum rod of fixed cross-section does not depend on how long the rod is. The ratio does depend on the diameter of the rod (How do you know this?), and you would probably guess that it depends also on the material of which the rod is made. For a rod of 0.635 cm diameter, you found a ratio of about 0.86 g/cm (its value depends somewhat on which particular alloy you used). We can scarcely escape the conclusion that 0.86 g/cm is purely a property of aluminum rod 0.635 cm in diameter. You can have an aluminum rod of this diameter any length you please, just as you can have a square of any edge-length or a circle of any diameter you please. The weight of such a rod, like the diagonal of a square and the circumference of a circle, may be any thing you please. Neither the length nor the weight of this size aluminum rod is a property of this size rod, for they may have any values at all. But the ratio of weight/length cannot have any value at all. Once you fix on aluminum rod, and once you fix its diameter as 0.635 cm, then you no longer have any choice in the matter: the ratio of weight/length is 0. 86 g/cm whether you like it or not. The quantity 0. 86 g/cm is therefore a property of this size aluminum rod, and not of what piece of the rod you happen to be talking about.

This property is often called "linear density", the ratio of weight/length for any material of fixed cross-section, whether it is platinum wire finer than a human hair or a giant steel girder weighing hundred of pounds per foot. Try to see the similarity among speed (say miles per hour), linear density (say grams per centimeter), and unit cost (say cents per pound). You can think of speed as the rate at which you accumulate miles behind you as you travel along; the speed is the number of miles accumulated in one hour. Unit cost is the rate at which your grocery bill piles up behind you as you throw pound after pound of pork chops on your grocery cart; the unit cost is the number of cents indebtedness accumulated per pound of pork chops bought. Linear density is the rate at which you use up grams of pencil as you feed the pencil into the pencil sharpener; the linear density is the grams of pencil ground up per centimeter of pencil fed in. Think of other "something per something" quantities and see how they all have a similar interpretation. You will have made a long step toward really understanding physical science if you can get a feeling for the real meaning of "X per Y."

If you have not already done so, make graphs for the data in both tables in Experiment 12. Plot weight vertically and length horizontally, placing both plots on the same graph. Label the two curves appropriately. Does the linearity of the curves confirm that weight is proportional to length?

Which curve is steeper, the one for the larger or the smaller rod? Can you relate the steepness of the curve to the magnitude of the proportionality constant? Which size of rod accumulates weight faster behind you as you run along its length? Which curve rises faster as you move to the right? Are the last two questions related? If you are shown a graph with two straight lines through the origin plotted on it, can you tell at sight which has the larger proportionality constant?

If a steel girder weighs 175 pounds per foot, how much will 12 feet weigh? If a glass tube weighs 0. 65 g/cm, how much will a piece 82 cm long weigh? If a copper wire 16 cm long weighs 0. 00144 gram, how much will 97 cm weigh? (Hint: first find out how much 1 cm weighs.)

7. Once Again, Lightly

When two variables, say X and Y, are so related that assigning a value to X automatically fixes the value of Y, then Y is said to be a function of X. The variable whose value you assign (X in this example) is often called the independent variable and the function (Y in this example) is often called the dependent variable.

To say that Y is a function of X implies that, given X, you can find Y. Any rule that tells you how to find Y when X is given is a representation of the function. Functions may be represented tabularly, graphically, or analytically. Each of these three has its own advantages and disadvantages.

It is often found that steadily increasing the value of the independent variable causes the dependent variable either to steadily increase or to steadily decrease; that is, causes the dependent variable to change always in the same direction. Such a function is called monotonic, and a monotonic function clearly may be either an increasing function or a decreasing function but cannot be both.

A special and very important kind of monotonic function occurs when Y and X are so related that Y/X always has the same value no matter how X (and Y) may change. In this case, Y is said to be proportional to X. Any of the following six statements is exactly equivalent to any other of them:

Y is proportional to X

$Y/X = k$ (where k is some constant)

$Y = k \times X$

$X/Y = 1/k$

$X = (1/k) \times Y$

The graph of Y vs. X is a straight line that passes through the origin.

The constant, k, in the above table is called the proportionality constant. The proportionality constant often has a simple physical interpretation, its meaning being, of course, dependent upon the meanings of the two variables, X and Y.

Further Classroom Discussion

To say that Y is a function of X does not tell you very much. It says only that (in general) the value of Y changes with the value of X but not how it changes. To say that Y is a monotonic function of X says a little more. Notice how each of the following statements says a little more than the last, until finally the last statement says it all:

Y is a function of X.
Y is a monotonic function of X.
Y is an increasing function of X.
Y is proportional to X.
Y = 7.12 times X.

If two variables, U and V, are connected so that U is proportional to V, does it follow that U is a monotonic function of V? If W is a monotonic function of Z, it is necessarily true that W is proportional to Z? Draw graphs to illustrate your opinion.

Consider the units of a proportionality constant. In the statement, "The diagonal of a square is proportional to the edge," we found that the proportional constant (which one?) is $\sqrt{2}$. We also found that this proportionality constant remains equal to $\sqrt{2}$ whether the measurements are made in inches, centimeters, or widgets. Suppose that the Hooke's law constant for a certain spring is measured to be 3.42 centimeters/gram. Would the units be the same if the measurements of extension and weight had been measured in inches and pounds respectively? Can you formulate a general rule telling when the numerical value of a proportionality constant does depend on the units used for the two variables and when it does not?

You might have noticed that most of the written matter in this book is explanation, questioning, discussion, or illustration of certain central points. There are a few sentences here and there, however, which are not of this nature, but are statements of the central points themselves. The whole book could be enormously reduced in size if all the discussion, explanation, and illustration were removed and only those statements retained which carry the meat of the points to be made. This attitude is very different from , say, a history textbook, where practically every sentence carries meat not contained in any other sentence. Most people would find it very difficult to understand a textbook on the basic principles of science if there were no explanations and illustrations and just-plain-talking-about the central points. You should learn to tell the difference between the meat and the dressing, however, and understand that many pages may be spent trying to make clear the meaning of a relatively few scattered central points. It is only these points that you are expected to learn, however; the rest is only to help you learn. For example the first section of this unit, beginning on page 56 and going all the way down to "Points to Discuss in Class" on page 58, contains only one sentence (or possibly two) that is really essential, All the rest is to help and prepare you to understand the meaning of that one sentence. Can you find this one central point?

Each of these central points is usually one of three possible kinds:

Definitions
Experimental Findings
Derived Conclusions

For instance, consider the sentence "When two variables are so related that their ratio is a constant, the variables are said to be proportional." This is a definition of the word "proportional." This is not something you are commanded to believe, for it contains nothing to believe. It is merely a signal to you that from now on we are going to use the word "proportional" in a certain way, and if you want to understand what we are talking about you had better learn the way we are going to use it.

Next, consider Hooke's law, which says "The extension of a spring is proportional to the weight attached." This, of course, is not a definition; it makes a direct statement, presumably of fact. But is this sentence something you are commanded to believe? No, because you carried out an experiment (you and thousands of other people) in which you gathered data that led you to the apparent truth of the statement. This then is an experimental finding, whose truth is discovered by experiment.

Finally, consider the statement "The extension of a spring is equal to the attached weight multiplied by a constant." This is not a definition either; nor is it directly an experimental finding, for what you found experimentally was that the ratio of extension to weight is a constant. But once you found that "E/W = k" (this was an experimental finding), then purely by logic you manipulated this equation to show that "IF it is true that E/W = k, THEN it is also true that E = W x k." Thus you are not commanded to believe this, but are led to see that "If the experimental finding is correct, then this derived conclusion is also correct." A derived conclusion is a statement whose truth follows logically from another statement. If you believe the first, then logically you must also believe the second, but you are not commanded to believe it without being shown why it is believable.

Physical science is like this throughout. You are never commanded to believe anything. If ever a forthright statement is made and you do not fully understand why you are expected to believe it, question it. Do not accept it unthinkingly, like an obedient puppy dog.

Unit IV

Weight, Volume, and Density

1. Measurements of "Amount"

Jerry was stringing a length of wire from one post to another in his back yard in order to make a rack to dry his raccoon skins. The posts were ten feet apart and he had only 9 1/2 feet of very thin wire. Seeing that he needed a larger piece of wire, he called to his brother.

"Tom," he asked. "I need a little more wire than this piece you gave me. Find me another bigger piece, will you please?"

"Coming up," Tom called, and a minute later he brought his brother a six-foot length of very heavy wire. Jerry looked at the piece in disgust.

"Can't you see that that piece is even shorter than the one I have?" he said. "I distinctly asked you to bring me a bigger piece of wire, and look what you brought me."

Now it was Tom's turn to be annoyed. "But this piece I just brought you is bigger than the one you have. It may be shorter, but it's a lot bigger. And you'd say so, too, if you weren't so mad."

"You're right," Jerry apologized. "You didn't know what I wanted it for, and since you didn't, I should have said I wanted a longer piece, not merely a bigger piece."

Tom and Jerry's little disagreement didn't turn into a fight, but serious arguments often result from the fact that two people are using the same word in different senses. The violent drama above resulted from Jerry's use of the word "big" to mean "long," while Tom's understanding of the word was quite different. What did Tom mean? There is no point in arguing that you should never say "big" when you mean "long", because the truth is that you won't often be misunderstood. But in scientific speech, one must always be careful to say exactly what he means, even to the extent of avoiding the use of words that are imprecise in meaning. Here is another example:

Suppose I have a block of wood and a block of iron. The wooden block is the size of a brick and weighs 1.5 pounds. The block of iron is the size of a half-brick and weighs 9 pounds. I set them before you and ask, "Is there a larger amount of wood in the wooden block than there is iron in the iron block?" Don't worry about trying to answer this question, because it cannot be answered. The reason it can't be answered is simply that it isn't a question, even though it looks like one! And the reason it is not a question is just that the word "amount" is not defined. If both blocks were iron, however, the question, "Is there a larger amount of iron in this brick-sized block than there is in this half-brick-sized block?" can be answered. In this case, we can regard the word "amount" as defined with suffieient precision, because all reasonable interpretations of the word would lead you to agree that the larger block contains the greater amount of iron.

Since, then, the word "amount" does not have sufficient precision for all our purposes, we must agree that we will not use it when there is any chance for confusion. If we had asked whether the iron or wood block had the greater volume, there would have been no difficulty; and if we had asked which has the greater weight, there would have been no difficulty. This is because volume is defined as a certain measurable geometric property of the block, and weight is defined as a certain measurable physical property of the block. "Amount" is a more general term of much less precision. (All this does not mean, however, that you should never use the word "amount. It is just that you must learn to avoid using it when it is not sufficiently precise for the purpose at hand.)

The present unit deals with two quantities that can be used to express amounts of matter. One is volume, the other is weight, and the sense in which we shall use the terms are indicated in the preceding paragraph, though they are not defined there. Definitions of weight and volume are extremely difficult to formulate, and we shall rely simply on your already having a good enough idea of what they mean.

If you have two different pieces of the same material then, you would expect that "the larger piece would have the greater weight." We have already seen, however, that this quoted statement is not very useful as a functional relation. A functional relation must involve two measurable quantities. It is true that weight is a perfectly definite measurable quantity, but what is meant by "largeness," or "size"?

Suppose you have a set of round sticks all the same diameter and all of the same material, but of different lengths. Would you say that "size" might be taken to mean "length", so that one could say "the weight of one of these sticks is an increasing function of its length"? You have already investigated this question in Experiment 12, and found that the weight of a stick of fixed material and fixed cross section is not only an increasing function of its length, but is in fact proportional to its length. In this case, "size" and "length" could be used interchangeably. Suppose you have a set of circular cylinders all the same length, but of different diameters. Would it be correct to say:

The weight of a cylinder is an increasing function of the size?

The weight of a cylinder is in increasing function of the diameter?

The weight of a cylinder is proportional to the size?

The weight of a cylinder is proportional to the diameter?

Let's try Experiment 13 and see.

Points to Discuss in Class

What do you notice this time about the ratios, weight/diameter? Allowing for experimental error, would it be fair to conclude that the weight of an aluminum cylinder of fixed length is proportional to its diameter?

What about the graph you made of weight vs. diameter? You have seen that, when two variables are proportional, their graph is a straight line through the origin. May you conclude from the graph that the weight and diameter are proportional? How does your answer to this question agree with your conclusion from the preceding paragraph?

Suppose that the spring you had used in Experiment 11 had been lost or damaged so that you had to begin Experiment 13 with a new spring whose spring constant you didn't know. Would it be neccessary to do Experiment 11 completely over again to determine its spring contstant? If the ratio for a given spring extension/weight, is the same for all weights (this is what you found in Experiment 11, did you not?), then how many measurements of "extension versus weight" do you need to determine the ratio?

If you have to make only one measurement of extension and weight to get the spring constant, then the function, extension versus weight, must be knowable from just one measurement. But this implies that the graph also is knowable from only one measurement, for the graph is only another way of representing the same function. Is one measurement, (that is, one point on the graph) enough to tell you the whole graph? Remember that, if two variables are proportional, their graph is a straight line through the origin and through the one point you can plot from the one measurement you made. How many straight lines can you draw passing through the origin and the one plotted point? A spring whose spring constant is known is said to be calibrated.

In the present experiment with aluminum cylinders of fixed length but different diameters, you found that weight is not proportional to diameter. Is the weight a monotonic function of the diameter? Notice again the important logical point that a monotonic function is not necessarily a proportional function, although a proportional function is necessarily monotonic.

Now that we have found that weight is not proportional to diameter in this case, we feel a little let down. It is one thing to find that weight is not proportional to diameter. It is quite another thing to answer the question: What function is it? In this case it is not difficult to find the answer. Go back to your data sheet for Experiment 13 and compute the square of the diameter for each line of the table. How many significant figures are you entitled to in these squares? What will be the units of these quantities? Enter their values in the second-last column of the table. Put a suitable heading in the blank space over the column, including the units. Now try working out the ratio, "weight divided by square of the diameter". What are the units of this ratio? Enter these new ratios for each line of the table, putting a suitable heading over the column, including units.

You should now be able to formulate a statement: "For aluminum cylinders of fixed length, the weight of the cylinder is proportional to ________".

We can write an analytical expression for this last statement in the form

$$W = k \times D^2$$

where W means the weight of the cylinder and D is its diameter. Now once you know k (which is constant as long as you are talking about cylinders of some fixed material and fixed length), you can always calculate W when you are given D. The value of k you found in Experiment 13 was about 5.39 g/cm^2; that is, $W/D^2 = 5.39$ when the weight is expressed in grams and the diameter in cm. Would the value of the constant still be 5.39 if the weight and diameter were expressed in ounces and inches instead?

Now if $W = 5.39\ D^2$, you can easily calculate W whenever D is given. How would W change if you double D? That is, you know that doubling D will cause W to increase; can you make a general statement about how much W will increase on doubling D? Suppose D has the value of d before doubling, and, of course, 2d after doubling. The weight, W_1, before doubling will then be $W_1 = 5.39 \times d^2$. The weight, W_2, after doubling will be $W_2 = 5.39 \times (2d)^2$. The ratio, W_2/W_1, then, is

$$\frac{W_2}{W_1} = \frac{5.39 \times (2d)^2}{5.39 \times d^2} = \frac{(2d)^2}{d^2},$$

the last fraction coming from the permissible cancellation of 5.39 from top and bottom. Now $(2d)^2$ means "(2d) x (2d)", doesn't it? And that means "2 x d x 2 x d." Thus we can write

$$\frac{W_2}{W_1} = \frac{2 \times d \times 2 \times d}{d \times d} = 4$$

Be sure you see where the 4 comes from. Then if $W_2/W_1 = 4$, it follows that $W_2 = 4 \times W_1$. That is, W_2 is four times as great as W_1. Therefore doubling the diameter will multiply the weight by four. This is true, of course, no matter what the diameter before doubling might be; because all we said was that d is the diameter before doubling, and we never committed ourselves to any particular value for d.

Does it surprise you that doubling the diameter does not merely double the weight, but quadruples it? If you think of the circular cross-section of the cylinder, this would mean that doubling the diameter of the circle quadruples the area of the circle, wouldn't it? Draw two 1-inch circles side-by-side and just touching each other on a piece of paper. Then draw a 2-inch circle whose center lies at the point of contact of the two small circles. Is the 2-inch circle more than twice as "big" as a 1-inch circle? Actually, its area is four times as great.

Do you see now that saying that "weight is proportional to size" for a certain kind of material (like aluminum) may or may not be true? If you have a bunch of sticks all of the same cross-section but of different lengths, it is certainly quite reasonable to refer to the length of the stick as its size; and in this usage of the word "size", the weight is in fact proportional to size. If you have a bunch of sticks all the same length but of different diameters, it is again certainly reasonable to refer to the diameter of the stick as its size; but in this usage of the word "size", the weight is not proportional to size.

If it sounds confusing to you that weight sometimes is proportional to size and sometimes is not, don't worry about it. It would be confusing to anyone. But you should see that the whole reason for the confusion lies in using the word "size" in two different meanings. If you avoid this ambiguous use of the word and replace it by "length" in the first case and "diameter" in the second, every bit of the confusion disappears. You then have that weight is proportional to length but weight is not proportional to diameter. Things are made very simple by the correct choice of words, aren't they?

Let's look into one more case. Do Experiment 14 now.

Points to Discuss in Class

What does the inconstant ratio, weight/diameter, tell you about the proportionality between weight and diameter? Does the curve of your graph agree with with this conclusion? Does the curve appear to be similar to the one you obtained in Experiment 13? The similarity between the two graphs suggests that perhaps the ratio, "weight/square of the diameter" might be constant here, too, as it was in Experiment 13. Try it, using the sixth column to record the quantity, $(\text{diameter})^2$, and the seventh for the ratio, $\text{weight}/(\text{diameter})^2$. How many significant figures are you entitled to in the ratio? Do you get a constant ratio this time?

Would it be correct to say that the weight is an increasing function of the diameter? Would it be correct to say that weight is an increasing function of the square of the diameter? Would it be correct to say that the weight is proportional to the square of the diameter?

If the weight is proportional neither to the diameter of the sphere nor to the square of the diameter, can you suggest something to try next? When your class agrees on what to try, do it, using the last two columns of Table I. What units do these quantities have?

2. Density

The last three experiments have shown you that when you are talking about pieces of aluminum, the weight of the piece is proportional to
- the length, when the pieces are rods of the same cross-section
- the square of the diameter when the pieces are rods of the same length but different diameters.
- the cube of the diameter when the pieces are spheres.

Or, if we say the same things symbolically, we could write

$$W_1 = k_1 \times 1$$

$$W_2 = k_2 \times d^2$$

$$W_3 = k_3 \times d^3$$

In these equations: W_1, W_2, and W_3 represent the weights of, respectively, rods of the same cross-section, cylinders of the same length, and spheres; and k_1, k_2, and k_3 are the corresponding proportionality constants. Nothing, either in these equations or in the corresponding word-statements above, tells you what the numerical values of the k's are; but you measured them in your experiments.

Now this is the kind of finding that causes a physical scientist to scratch his head and pace the floor, or at least to squirm in his chair. Here we have some pieces of the same material, all aluminum, and the weight of the piece varies in a crazy way with the size, being sometimes proportional to some dimension, sometimes proportional to the square of some dimension, and sometimes proportional to the cube of some dimension. Isn't there some way we can unify all these findings into a single larger idea, so that we don't have so many diverse individual ideas separately to remember? One of the main goals of physical science is to find such unifying ideas. Let's try it in this case.

The thing that strikes us as possibly unifiable here is this: All the metal pieces were of aluminum, and there should therefore be an underlying sameness about the three functional relations. Still, there are three different k's, each of which had to be separately measured. Might there not be a way to relate one k to another, so that you would have to make only one measurement for aluminum, and then all the k's would follow from that one measurement? This might be possible, so let's think some more about it.

Suppose I have a piece of aluminum of a certain weight. Then in my imagination I add another piece of aluminum to it; the weight of the piece will of course increase. But let me add this second piece in a special way. I will add it to a "rod of fixed cross-section" in such a way that the augmented rod is still of the same cross-section but a little longer. Or, I will add the same piece to a "cylinder of fixed length" in such a way that the augmented cylinder is still the same length but a little larger in diameter. (I can do this by "coating" the added piece like a sheet of wrapping paper around the curved surface of the cylinder, but not on the flat ends.) Or, I will add the same piece to a "sphere" by buttering it uniformly over the entire surface of the sphere so that the augmented piece is still a sphere only a little larger.

Now since the piece I added was each time the same piece, whether it was added to rod, cylinder, or sphere, it is clear that the weight of the piece must increase the same amount each time, no matter what the shape we started with. This suggests that what really counts in determining the weight of a piece of aluminum is the volume of aluminum contained in the piece. Surely, you say, you would find that the weight of a piece of aluminum is an increasing function of its volume. Perhaps we could investigate this guess, and find out whether it's true; and if we're lucky, perhaps we could even find out exactly what increasing function it is.

Now do Experiment 15, after which we will have more

Points to Discuss in Class

Did you find that the weight of a chunk of aluminum is an increasing function of its volume? Did the constancy of the ratios and the linearity of the graph show that weight is proportional to volume? Suppose that, after you had finished with the eight blocks you used in this experiment, you had been given an invisibly small piece of aluminum as your ninth block. The weight and volume would both be zero of course. You therefore could not compute the ratio, weight/volume. On the other hand, you have learned that you may call the ratio, 0/0, anything you please. What would you like to call it in this case?

In analytical form, we find by experiment that

$$\frac{W}{V} = k,$$

where W is the weight of the piece, V is its volume, and k is some constant. What are the units of this constant when W is in grams and V in cc? Would the numerical value of k be different if we measured the weight in pounds and the volume in gallons? The value for k is about 2.7 g/cc, depending somewhat on what particular aluminum alloy you used.

Do you understand the meaning of the statement, "W/V is a constant for all pieces of aluminum"? The weight and the volume of aluminum chunks are variables. You may have a chunk of aluminum of any weight you choose; you may have a chunk of aluminum of any volume you choose. But you cannot choose both. Once you have fixed on some certain volume for a chunk of aluminum, the weight is fixed whether you like it or not. You have all the freedom you wish to choose either the volume or the weight, but you cannot choose the ratio of weight/volume.

The thing to notice in the last sentence above is that you cannot choose the ratio. There always is a ratio, of course, but its value is "chosen for you". The aluminum itself, so to speak, does the choosing of the ratio, and your experiment shows that it always chooses the same ratio. Another way to put the point is to remind you of what you found in Experiments 6 and 8. You found that the ratio of $\sqrt{2}$ for diagonal/edge for a square is purely a property of being square and not on what square you are talking about. You found that the ratio of π for circumference/diameter for a circle is purely a property of being circular and not on what particular circle you happen to be talking about. Now in this experiment you found that the ratio of 2.7 g/cc is purely a property of being aluminum, and not on what piece of aluminum you're talking about. The weight of a certain piece of aluminum is not a property of aluminum, for the weight depends on what chunk you are dealing with. The same is true of volume. But the ratio of weight/volume is the same for all pieces of aluminum, does not depend on what piece of aluminum you measure, and is purely a property of being aluminum.

The ratio of weight /volume for any kind of material is called the density of the material. From now on we shall use d instead of k to symbolize the ratio, and we can write

$$\frac{W}{V} = d$$

This equation may be taken as the definition of density. From it you should be able to derive mathematically that

$$W = d \times V \quad \text{and} \quad V = W/d$$

You should not bother to try to memorize these last two equations. You must, of course, memorize the defining equation (if you expect to remember it), but it is foolish to memorize the other two because they are so easily derived from the defining equation.

You should try to get a feeling for the meaning of the quantity called density. Try to see the close analogy in meaning among density and, say, speed and unit cost. Speed is the rate of piling up distance as time goes on -- say miles covered per hour traveled. Unit cost is the rate of piling up your grocery bill as you but more hamburger -- say dollars of grocery bill per pound of hamburger. Density is the rate of piling up weight as more and more volume is added -- say grams of weight accumulated per cc of volume added.

What is the weight in grams of one cc of a material whose density is d? Let W_1 be the weight of one cc of the material. Its volume, of course, is 1 cc. According to the definition, then, density = weight/volume = $W_1/1 = W_1$. That is, $W_1 = d$. In words, the weight of one cc of the material is numerically equal to the density. Otherwise stated the density (in g/cc) of a material is simply the weight (in grams) of one cc of the material. Do you see that this statement is merely another way of wording the last part of the last sentence of the preceding paragraph?

You should be able to compute the volume of a piece of aluminum if you know its weight and you should also be able to compute the weight if you know its volume. Let us take a look at two such problems.

First, what is the weight of 17.6 cc of aluminum, given that the density is 2.71 g/cc? There are many equivalent ways to work this problem, differing mostly in the thought processes used to arrive at the required arithmetic. The worst way is to substitute in the formula, saying something like this: we are given that V = 17.6 cc and that d = 2.71 g/cc. From one of the above formulas, we know that W = d x V. Then W = 2.71 x 17.6 = 47.7 grams, which is the correct answer.

Now you should ask: If this method gives the correct answer, why would anyone call it the "worst" way? If the method works, what is wrong with it? The main answer is that blind unthinking substitution in a formula deprives you of a chance to think. If you baby yourself by working problems always by a recipe, then you deprive yourself of the chance to acquire a real and comfortable understanding of the ideas. Refuse to baby yourself; force yourself to think. Then one day when you have to think, it won't be a stranger to you. Noone hires a scientist because he knows a lot of formulas or because he can substitute numbers in formulas. That's what encyclopedias and computing machines are for! Whether you are going to be a scientists, a housewife, a baseball player, farmer, or salesman, you will have to learn to think. Now is the time to start.

Let's do the same problem by thinking; it's extremely easy! You say to yourself: I am asked to find the weight of 17.6 cc of aluminum. (I'm not going to find the answer just by sitting in my chair and waiting for someone to tell me the asnwer. I probably won't be able to find the answer by looking it up in a book, because the chances are slim that anyone has ever worked out exactly this problem before. I don't want to ask someone else the answer, because I want to be the kind of person that other people ask, not the kind that has to ask other people. I have no recourse but to work it myself.) I could work out the weight of 17.6 cc if I knew the weight of one cc, because the weight of 17.6 cc is evidently just 17.6 times the weight of one cc. (How do you know this?) Now I'm given the density as 2.71 g/cc; what does that mean? Why that means that aluminum weighs 2.71 grams per cc; that is, each cc weighs 2.71 grams. Well if one cc weighs 2.71 grams, how much does 17.6 cc weigh? That's all there is to it!

Another way to approach this problem is useful to know about because the same idea can be used to work much more complicated problems where even the best "thinkers" might get lost. In this procedure, one thinks only of the units.involved. We are given 17.6 CC and asked to find the number of GRAMS. We are asked to go

From	To
cc	grams.

You now ask yourself: according to the rules for working with units, how can I "change" cc into grams? The first thing you must do is put in "grams" where you don't have grams. You can do this by multiplying by grams:

cc x grams gives cc-grams

according to the rule on page 35. Thus multiplying cc by grams would give us cc-grams, which still isn't what we want but at least it has "grams" in it! We still have to get rid of the unwanted cc that occurs in cc-grams. Suppose we divide what we now have by cc. We would then have

$$\frac{\text{cc} \times \text{grams}}{\text{cc}}$$

But you recognize now that we are multiplying and dividing by "cc", and we may therefore cancel them out:

$$\frac{\text{cc x grams}}{\text{cc}} \quad \text{gives} \quad \text{grams}$$

We can rearrange the thing on the left as follows

$$\text{cc x } \frac{\text{grams}}{\text{cc}} \quad \text{gives} \quad \text{grams}$$

and then we have what we want: In order to convert "cc" to "grams", we have to multiply by a quantity whose units are grams/cc. But those are the units of density. Hence to "convert" a volume (in cc) to a weight (in grams), you must multiply by density (g/cc). The arithmetic now follows immediately: you must multiply 17. 6 x 2. 71 to get the answer in grams. Easy, isn't it?

Now let's work another problem: What volume would 43. 9 grams of aluminum occupy, if the density is 2. 71 g/cc? Given the weight, find the volume. Of course, one way to work the problem is to substitute the given numbers in the formula (page 90). But this is the baby way and you prefer to use the thinker's way! Let's see whether we can think it out.

One way is to lean on what you know of arithmetic. You say to yourself: I have a block of aluminum that weighs 43. 9 grams. The density of aluminum is 2. 71 grams/cc, which means that each cc weighs 2. 71 grams. In my block of 43. 9 grams, then, every cc of it weighs 2. 71 grams. The number of cc's in the block then is the number of (2. 71 grams)'s in it. That is, how many times is 2. 71 contained in 43. 9? Thus the volume is 43. 9/2. 71 or 16. 2 cc.

Another way is to pretend that you already know the answer and use symbols. Suppose we call the unknown volume, V. Now if I have a block of aluminum whose volume is Vcc, and each cc weighs 2. 71 grams, then the weight of the whole block is 2. 71 x V grams. But the weight of the whole block is also given as 43. 9 grams. Hence

$$2.71 \times V = 43.9.$$

From this you should easily be able to show that $V = 43.9/2.71$ cc.

Still another way is to think only of the units. We are given grams and we wish to find cc: How can we go

From	To	
		?
grams	cc	

Using the same reasoning we used before, you can see that we could convert grams into cc by multiplying grams by cc/gram, for then we would have

$$\text{gram} \times \frac{\text{cc}}{\text{gram}} \quad \text{gives} \quad \text{cc}$$

because you may cancel out the "gram" upstairs and downstairs. Now you know from your study of the arithmetic of fractions that multiplying by a fraction is the same thing as dividing by that fraction turned upside down. (If you don't know this, please pretend that you believe it for a moment and we'll prove it in the next paragraph.) Therefore, multiplying by cc/gram is the same as dividing by gram/cc. That is

$$\frac{\text{gram}}{\text{gram/cc}} \quad \text{gives} \quad \text{cc.}$$

This last statement says that dividing the weight (grams) by the density (cc) gives the volume. That is, the volume is 43.9/2.71, same as before.

Now, if you did not see why it's true that multiplying by a fraction is the same as dividing by the fraction turned upside down, think of it this way. Suppose that we wanted to multiply any number, A, by and fraction, B/C. Say the answer is P. Then

$$A \times \frac{B}{C} = P.$$

Now multiply both sides of this equation by C/B. Then

$$A \times \frac{B}{C} \times \frac{C}{B} = P \times \frac{C}{B}.$$

Now the left-hand side of this equation is merely A because we can cancel out the B's and C's that appear upstairs and down. Then we have

$$A = P \times \frac{C}{B}.$$

Now divide both sides by C/B. Then

$$\frac{A}{C/B} = \frac{P \times C/B}{C/B}$$

But on the right-hand side, we are both multiplying and dividing by C/B; hence they can be canceled:

$$\frac{A}{C/B} = P.$$

But now if you go back, you will see that P was originally defined as A x B/C. Therefore you have shown that $\frac{A}{C/B}$ is the same thing as A x B/C. QED

[When you turn a fraction upside down, the new fraction is called the reciprocal of the other. Like brothers, if M is the reciprocal of N, then N is the reciprocal of M. (Do you see why?) We have shown that multiplying by a number is the same as dividing by its reciprocal. You should be able by yourself to show that dividing by a number is the same as multiplying by its reciprocal.]

Before leaving the numerical problems we just worked out, there is something that ought to be called to your attention. Notice that we did not work these problems by some set routine method that somebody told us to use. By using logic, we worked out our own methods. We therefore know they have to give the correct answer without our needing someone to tell us so. IT IS FAR, FAR MORE IMPORTANT THAT YOU SEE HOW TO WORK THESE PROBLEMS THAN THAT YOU MERELY GET THE RIGHT ANSWER. IT IS FAR MORE IMPORTANT THAT YOU UNDERSTAND HOW WE REASONED OUT THE METHODS THAN THAT YOU MEMORIZE THE METHODS AS RECIPES. Keep in mind that you can work out your own method to solve the problem and do not need formulas or someone else to tell you how. Of course you might need help in the beginning; the point is that a proper way to solve a problem is decided by logic, not by someone's authority.

3. A Unification

Do you remember that we left our friend, the physical scientist, scratching his head and pacing the floor, way back on page 88? Well, now we are in a position to help the poor fellow. You remember we had exhibited some experimental findings in this way: If we let

W_1 mean the weight of an aluminum rod 0.635 cm in diameter but of variable length, L;

W_2 mean the weight of an aluminum cylinder 2.54 cm long but of variable diameter, D; and

W_3 mean the weight of an aluminum sphere of variable diameter, D;

then the results of Experiments 12, 13, and 14 could be summarized in the functional relationships:

$$W_1 = k_1 \times L \qquad \text{where } k_1 = 0.855$$

$$W_2 = k_2 \times D^2 \qquad \text{where } k_2 = 5.39$$

$$W_3 = k_3 \times D^3 \qquad \text{where } k_3 = 1.414$$

(The proportionality constants as you found them in your experiments are already entered here. These numbers may not be exactly the same as yours-they depend somewhat on the particular alloy you used-but yours should have been close to these.) Our head-scratching, floor-pacing physical scientist now was wondering to himself like this: Here I determined three different proportionality constants. All are concerned with weights of aluminum blocks of certain specified shapes. Surely, since all the blocks are of the same material, these three different proportionality constants are somehow related. What is the relationship among them? And then our physical scientist starts thinking.

Suppose I forget for the moment that I have already measured the proportionality constant, k_1, between weight and length of aluminum rods 0.635 cm diameter. Instead, let me work out the weight of such a rod from the known density of aluminum. To do so, I would have to find the volume of aluminum in the rod and multiply it by the density:

$$W_1 = d \times V \tag{1}$$

The density, d, I know; what about the volume, V? Well, these rods are cylinders, and I can always find the volume of a cylinder from the geometric formula

$$V = \frac{\pi}{4} \times D^2 \times L$$

where D is the diameter of the rod and L is its length. Now this last equation says that $\frac{\pi}{4} \times D^2 \times L$ is another name for V. Therefore I may replace the V in equation (1) above by its other name and obtain

$$W_1 = d \times \frac{\pi}{4} \times D^2 \times L \tag{2}$$

Now look at this last equation carefully. We are talking exclusively about aluminum rods of just the one diameter, 0.635 cm. In this equation, then, d is a temporary constant, being the density of aluminum; D^2 is a temporary constant, being the square of 0.635; and of course $\pi/4$ is an absolute constant whose value you can work out. Since d, D^2, and $\pi/4$ are all constants, if you multiply them together, there is only one product you can get; that is, their product is a constant. That is,

$$W_1 = (\frac{\pi}{4} \times d \times D^2) \times L. \tag{3}$$

This equation is identical with equation (2) except that the first three factors have been lassoed together in parentheses to emphasize that all together they are simply one constant. Now if you compare equation (3) with the first equation displayed on page 94, you will immediately see that k_1 is simply another name for $\frac{\pi}{4} \times d \times D^2$. Since you know, or can easily work out, the numerical values of $\pi/4$ and d and D^2, you should now compute the value of k_1 and see how closely it agrees with the value you obtained for the proportionality constant in Experiment 12.

Remember that d and D^2 both have units (what are they?), while $\pi/4$ is without units. Then, applying the rule for units when multiplying, what are the units of $(\frac{\pi}{4} \times d \times D^2)$? What did you get for the units of k_1 in Experiment 12? You have now determined k_1 in two ways: experimentally in

Experiment 12, and now theoretically. Do the two determinations agree both numerically and with respect to their units?

Now let's tackle the second proportionality constant, k_2, relating weights and diameters for aluminum cylinders of length 2.54 cm. Of course equation (1) still applies, with the change that W_1 now becomes W_2. But we are also again talking about cylinders, so that the formula for volume remains as before and we can use equation (2) with W_1 changed to W_2:

$$W_2 = d \times \frac{\pi}{4} \times D^2 \times L \qquad (4)$$

The quantity $\pi/4$ is still a constant, of course, and since we are still talking about aluminum, so is d a constant. With the other factors, however, there is a difference. This time we are talking about cylinders of fixed length and variable diameter, so that L is a constant but not D^2. Thus we can rearrange the right-hand side of (4) and lasso quantities as follows:

$$W_2 = \left(\frac{\pi}{4} \times d \times L\right) \times D^2 \qquad (5)$$

Here again, the quantity in parentheses is, all together, a single constant. If you compare equation (5) with the second equation displayed on page 94, you will immediately see that k_2 is merely another name for $(\frac{\pi}{4} \times d \times L)$. Since you know the numerical values of d and L, you should be able to work out the value of k_2. Do it, and see whether the k_2 you get by this theoretical method agrees both in numerical value and units with the value you obtained from Experiment 13.

You ought now to be able to compute k_3, the proportionality constant relating weights of aluminum spheres to their diameters. Notice that equation (1) still applies (change W_1 to W_3, of course) and recall that the volume of a sphere is $\pi D^3/6$. Compare the computed k_3 with the experimentally measured value you obtained in Experiment 14. Do they agree in both numerical value and units?

So you see that the three proportionality constants, k_1, k_2, and k_3, are closely related after all. The main feature that makes physical science such a pleasing study is the continual recurrence of unifying ideas like this one, unifications that can be thought out just by the power of logical reasoning.

4. Densities of Various Solids

Now you understand that the density of aluminum is an intrinsic property of aluminum in the sense that every piece of aluminum in the world has always the same density. One of course feels that the same should be true of any other material that can be definitely specified. You should now do Experiment 16, which is concerned with the determination of the densities for several other materials.

Points to Discuss in Class

The densities you measured ranged from a low of about 0.6 g/cc for wood to a high of about 11.3 g/cc for lead. (The exact values will vary because there are different kinds of wood and plastic, different alloys of lead, brass, and steel.) How is this variance in density reflected in the graphs of weight vs. volume for the five materials?

Do all five curves have the same slope? Recall that one way to think of density is as "the rate at which weight is accumulated as you add more volume to the pile." If we add more volume to a pile of lead and also to a pile of wood, which pile will have its weight increased the more for each cc added? As you move to the right on these curves, you are increasing volume, are you not? Which curve rises more rapidly as you move to the right? Do you then feel *why* it is that the greater the density of the material, the steeper is its curve, weight vs. volume?

For any material, a piece of zero volume of course has zero weight. Thus the origin (where weight and volume are both zero) must lie on the curve for weight vs. volume for every material. That is, the curve of weight vs. volume must always pass through the origin, for any material at all. If the curve is known to be a straight line, how many other points do you need in order to draw the curve? How many pieces of a material must you measure (weight and volume) in order to determine the density? If the ratio, weight/volume, is always the same, how many pairs of weight and volume must you measure in order to know the ratio for all weights and volumes? Do you see the interconnection among these last three questions?

You are given that the density of brass is 8.4 g/cc and the density of steel is 7.7 g/cc. See whether you can answer the following questions:

Which is the heavier, a block of brass or a block of steel, if they both have a volume of 1 cc?

Which is the larger volume, 1 cc of brass or 1 cc of steel?

Which has the larger volume, 1 gram of brass or 1 gram of steel?

Two cylinders, one brass and one steel, are both 3 cm in diameter and both weigh ten grams. Which is longer?

Which is heavier, a one-gram block of brass or a one-gram block of steel?

It is desired to make a metal block measuring 2 cm x 3 cm x 4 cm, weighing not more than 200 grams. Can this be done with brass or steel, neither or both?

Will it take a greater weight of brass or of steel to make a statuette whose volume is 22 cc?

Will it take a greater volume of brass or of steel to make a miniature baseball bat weighing 10 grams?

Which has the greater density in pounds per cubic inch, brass or steel?

Suppose that you can make weighings with your spring that are good to 0.01 gram. If you weigh a block of metal whose weight is just more than 1 gram, how many significant figures would you be entitled to in the weight? If you weigh a block whose weight is just more than 10 grams, how many significant figures in the weight? If you knew the volume of the sample accurately to four significant figures, how many significant figures would you be entitled to in the density of the 1-gram block? In the 10-gram block? If you wanted to obtain the highest precision possible using your apparatus to determine the density of aluminum, would you choose to make your measurements on a small or large piece of metal?

Which is heavier, wood or lead? You would probably answer lead, of course. Yet you know that 10 poinds of wood is certainly heavier than 1 pound of lead. What do you mean when you say "lead is heavier than wood"? Notice that we commonly use the word "heavy" in two quite different senses: in one sense we use the word to mean "having a great density." There is nothing wrong with this double use of the word as long as you are aware of possible confusion and avoid it when you should. It is this double meaning of the word "heavy" that forms the base for the riddle "Which is heavier, a pound of lead or a pound of feathers?" Either "lead" or "neither" is the correct answer, depending on which meaning of the word "heavier" the questioner has in mind. It is nonsensical to spend hours arguing over the "correct" answer, when the real point is "What does the question mean?" Once it is settled what the question means (and any good dictionary will give both meanings for the word "heavy"), there is no longer any argument. Many passionate arguments are the result of unagreed meanings of words. You should learn to recognize this fact and guard against it.

Since different materials have different densities, it ought to be possible to use the property of density to identify an unknown material. Experiment 17 is a detective game based on this idea. Do it now!

Points to Discuss in Class

How did you come out in your identification?

If you were to rub off the paint on the two blocks whose densities agreed with none of those you measured in Experiment 16, you would find them to be brass. Can you explain the discrepancy? It is not a different kind of brass.

It is also entirely possible for two different materials to have the same density -- aluminum bronze and nickel steel might be examples. It is also entirely possible that two different materials might agree in density to the second decimal place but disagree beyond that. Do you see why one has to be careful when he uses only one property to decide identity?

Suppose you had two samples of material that agreed in density, but you were not assured (as in this experiment) that they are each one of five materials. Could you safely conclude that they were the same material? Suppose in addition that they both had the same color, taste, and hardness. Could you then safely conclude that they were the same material? How many properties must coincide before you can certainly say that two samples are the same material?

5. Density of Liquids

Nothing in the definition of the term density prevents its application to liquids. It is perfectly meaningful to speak of the density of a liquid, because one can measure both the volume and the weight of a sample of liquid and then compute the ratio. Do Experiment 18 now, which involves the measuring of the densities of some liquids.

Points to Discuss in Class

Of the four liquids, which has the greatest density and which the least?

Water does not mix with either benzene or carbon tetrachloride. If you placed a few drops of water and a few drops of benzene together in a tube, what would you expect to happen? Try it, and tell which layer is which, and why. Close the tube with the thumb, shake it violently, let it stand a minute, and observe what happens. Can you explain? Do the same experiment with water and carbon tetrachloride. Now which layer is which? If two liquids do not mix, can you tell from their densities which will float on top?

Pipette once cc of carbon tetrachloride into a tube, then one cc of water, then one cc of benzene (in that order). Explain what you observe. Close the tube with your thumb, shake it violently, let it stand a minute, and observe. Can you explain what happened this time?

Refer to the densities of water and alcohol as you determined them. Which would you expect to float on top if they were placed together in a tube? Try it, and then explain what you observe.

Tom and Jerry go to the drugstore, Tom to buy a pound of benzene and Jerry to buy a pound of carbon tetrachloride. The druggist gives each of them a full bottle, but Jerry's is much smaller than Tom's. Why?

Did you notice that in this experiment we used the "No-load position" of the spring as the position with the vial hanging on it, whereas the spring was calibrated with truly "no load"? Does this bother you? It should! You calibrated the spring and found the weight hanging on it to be proportional to the length the spring extends beyond its length when nothing hangs on it. Suppose that you were to calibrate it again, this time first hanging a bucket on it (thereby giving the spring an initial extension) and then looking for the functional relationship between "weight added to the bucket" and "extension of the spring beyond what thebucket extends it." The questions come up: What right have you to suppose that these new variables are proportional? And even if they are, would the proportionality constant be the same?

The answers to the questions are: "Yes, the new variables are proportional and the proportionality constant is the same." The procedure you used in Experiment 18 is valid, even though you calibrated the spring without the bucket. But PLEASE, you are not to accept someone's word for it. You have a duty to ask why the procedure is valid. Here is why:

When you calibrated the spring, you found that

$$E = k \times W \qquad (1)$$

where E is the extension of the spring beyond where it hangs with no load at all, W is the weight hanging on it, and k is a constant as long as we are dealing with that particular spring. Now suppose you are interested in the weight of a certain pay load, W_p. We could find the value of W_p by hanging it by itself on the spring, observing the extension (call it E_p), and then calculating W_p from the equation

$$E_p = k \times W_p \qquad (2)$$

as you have now done so many times. (Here, notice that E_p is the extension beyond no load at all that you would get if you attached the pay load by itself.) Suppose, however, that the pay load is a liquid that you cannot hang on the spring by itself. You therefore put it in a bucket, and hang both the bucket and the pay load on the spring. Let us call the total weight (bucket plus pay load), W_T, and the extenstion it causes (beyond no load at all), E_T. Equation (1) still applies of course, for the W and E in that equation mean total load and extension beyond no load at all. Therefore

$$E_T = k \times W_T. \qquad (3)$$

Now the total load, W_T, is made up of two parts. W_p, the pay load, and W_B, the weight of the bucket. That is

$$W_T = W_p + W_B$$

This means that "$W_p + W_B$" is another name for W_T, and we may replace the W_T that appears in (3) by this new name. Equation (3) then looks like this:

$$E_T = k \times (W_p + W_B). \qquad (4)$$

But now you know that if you multiply the sum of two numbers by a multiplier, you get the same result as if you multiply each of the numbers separately by the multiplier and then add. That is

$$k \times (W_p + W_B) = (k \times W_p) + (k \times W_B).$$

We may then replace the right hand side of (4) by its new name given by the last equation:

$$E_T = (k \times W_p) + (k \times W_B). \qquad (5)$$

Now consider what happens if you hang only the bucket on the spring. Equation (1) still applies, of course, and we know that the bucket will extend the spring an amount proportional to its weight. If we call E_B the extension beyond no load at all produced by the bucket, then we know that

$$E_B = k \times W_B. \qquad (6)$$

Let us subtract E_B from E_T: we get, of course, $E_T - E_B$. But equations (5) and (6) give us two other names for E_T and E_B. We can use these two names instead and write

$$E_T - E_B = (k \times W_p) + (k \times W_B) - (k \times W_B).$$

Look at the right- hand side of this equation. It tells you to take the number $(k \times W_p)$ and then add to it $(k \times W_B)$, and after you have done that, to subtract $(k \times W_B)$ away again. You know that adding and subtracting the same thing to any number leaves the number unchanged. (Does this remind you of the numerical property that dividing and multiplying by the same number leaves things unchanged?) Therefore the last equation above could be written

$$E_T - E_B = k \times W_p.$$

If you now look at the right-hand side of this equation and the right-hand side of equation (2), you will see that $(k \times W_p)$ and E_p and $(E_T - E_B)$ are all just different names for the same quantity. That is,

$$E_p = E_T - E_B.$$

This last equation is what we are looking for: it says that the extension E_p that the pay load would produce by itself (if you could attach it) is simply (the total extension it would produce when added to the bucket) minus (the extension produced by the bucket alone). This is what we wished to prove, and we now know that treating the position of the spring with bucket attached as a "no load" position is an entirely valid procedure.

You found the density of water to be about 1.00 g/cc and of alcohol about 0.79 g/cc. You now know that alcohol and water mix together. What would you expect the density of a mixture to be? This question is looked into in Experiment 19, which you should now do.

Points to Discuss in Class

Since water and alcohol have different densities, yet mix together completely, you can see that a mixture of the two could not have the same density as both pure materials, for the mixture could have only one density. Your intuition would tell you to expect that the density of the mixture would depend on its composition, for a mixture with only a little alcohol in it would have a density nearly the same as water, whereas a mixture that is mostly alcohol would have a density nearly the same as alcohol. Thus gradually adding alcohol to water would have to bring the density all the way down from about 1.00 g/cc eventually to about 0.79 g/cc. The principle of continuity suggests that this change would be a gradual one, with out big jumps in it.

Do your experimental findings indicate that the density is a function of composition? You have exhibited this function in two ways -- tabularly and graphically. Is density a monotonic function of composition? Is it proportional?

Since you cannot write an equation "$d = k \times c$" (where d is the density and c is the composition), the question comes up: Can we write an analytical representation of this function in some other way? It is not easy to answer this question. A physical scientist would feel that there must be some analytical expression connecting density and composition, but the truth is that physical science has not yet progressed to the point where we can say just what that expression is. We must therefore be satisfied with the graphical and tabular representations. At the same time, however, most physical scientists feel that this limitation is only temporary and that eventually such an analytical expression will be worked out. It will probably be very complicated. Lesson: physical science is incomplete; not everything in its domain is understood.

This may surprise you. Physical scientists are able to work out problems of seemingly vast complexity, like say the paths of the planets around the sun. Yet they cannot work out a problem of seeming simplicity like the density of a mixture of alcohol and water. Why is this? The answer is simply that the density problem is only "seemingly" simple. The astronomical problem can be solved by representing a dozen or so bodies by a dozen spheres that attract each other. Complex as this problem turns out to be, it can be handled; and in fact it is enormously more simple than the alcohol-water mixture, which must be treated as a collection of millions of molecules that attract and interfere with each other, and have complicated and even changing shapes. The astronomical problem only seems more complex than the alcohol-water problem; probably because it deals with a physically large system that you must look at from a distance while the other is so physically small that you can hold it in your hand.

Suppose you added 10.00 cc of water to 10.00 cc of alcohol. What would be the volume? You may be tenpted to add the numbers and answer 20.00 cc. But are you sure that "adding the two volumes of liquid together in a test tube" is equivalent to "adding the two numbers together by arithmetic." Is this a case where "putting together" does not mean "adding"? You ought to be suspicious about this! Adding 10 cc of water to 10 cc of water does indeed give 20 cc; the same is true if both samples are alcohol; but do you really have a well-founded reason to believe that it also is true when one sample is alcohol and the other water?

Well you don't have to sit and argue about it! You cancalculate it from data you now have. Since you know the densities of pure alcohol and pure water, you cancalculate the weights of 10.00 cc of water and of 10.00 cc of alcohol. Do it. Refer to the table at the bottom of the second work sheet for Experiment 10. You then can add these weights together to get the total weight, and you can divide the weight of alcohol by the total weight to get the composition (fraction of alcohol). Now you can look on your graph to see what is the density of a mixture of this composition. Look up this density, and then knowing the density and the total weight of the mixture, you can calculate its volume. How does the actual volume compare to the sum of the individual volumes?

If you had a mixture of alcohol and water and you wanted to know what percentage of the solution is alcohol, how would you go about analyzing it?

6. Concentration

Tom and Jerry were visiting some friends in New York City, and they found it very different from their small home-town in Texas. Packed like sardines in a can, they were riding on one of the subways.

"Whew!", Jerry exclaimed, trying to make a little room so he could move his arms. "I never saw so many people in my life. How many people are there in New York City?"

"About 8 million," Tom replied, "and I think most of them are on the subway with us."

"Well, I'm glad there aren't that many people in Texas," Jerry sighed. "I like the open spaces. Say, how many people are there in Texas, anyway?"

"About 8 million, same as in New York City."

"Now wait a minute," said Jerry in surprise. "I've never seen anything like this at home. If there are as many people in Texas as in New York City, how come it's so easy to move around at home? People in Texas aren't nearly so crowded as they are here in New York City. You must be wrong about the populations you told me."

Was Tom wrong with his figures? No, actually the population of Texas is about the same as that of New York City, yet one gets the impression of far more people on the average in New York than at most places in Texas. The point is that crowdedness of people and number of people are two different things. Ten people in a telephone booth would be rather crowded, but ten people in a football stadium might be so far apart that they couldn't converse comfortably. Question: is "crowdedness" a quality capable of numerical expression, or is it another of those qualities where you only "feel" a difference?

If you think about it a moment, you will realize that crowdedness refers to the number of people packed into a given space. Suppose we take the "given space" to be one square mile. The area of New York City is about 400 square miles and of Texas about 250,000 square miles. Can you calculate the number of people on the average in one square mile? Do so, and you will find there are 20,000 people per square mile in New York City but only 32 people per square mile on the average in Texas. No wonder you notice a difference! Notice that you obtained the crowdedness by dividing the number by the space they occupy.

Another word for crowdedness is "concentration." You can talk about the concentration of many different things. For instance, you could speak of the concentration of sugar in a sugar syrup, of salt in different samples of salt water, of acetic acid in different samples of vinegar, and so on.

If I gave you two samples of sugar syrup -- one thin and watery and almost tasteless and the other thick and sweet --, and asked you which contains the more sugar, you might answer, the thick one. But if I gave you a whole tank car full of the thin syrup and only a thimble of thick, you would have to agree that the thinner one actually contained the more sugar.

The confusion here is similar to that cited on page 98 about whether wood or lead is heavier. You remember we saw the root of that puzzle as the ambiguity in meaning of the word heavy; sometimes it refers to weight and sometimes to density. With the sugar solutions, too, there is an ambiguity. When I ask which syrup contains the more sugar, do I mean actual amount of sugar or do I mean concentration? The proper answer, perhaps, is that the tankcarful of thin syrup contains more sugar. If I had asked "which has the greater concentration of sugar?", the answer of course would be the thimbleful of thick syrup.

Whatever the stuff whose concentration you are talking about, the definition of concentration is:

$$\text{Concentration of Stuff} = \frac{\text{Amount of Stuff}}{\text{Space it occupies.}}$$

You can express both amount and space in lots of different ways, and therefore express concentration in lots of different ways. From the rule regarding units when dividing quantities, you can see that the units of a concentration are always "something per something." Here are some examples:

People per square mile

Monkeys per barrel

Grams per cc

Pounds per gallon

Parts per million

Ten per cent

Notice that density is itself a kind of concentration. It measures the actual weight of matter packed (crowded) into a unit of volume.

Your intuition tells you that the concentration of black jelly beans in a one-pound box of different-colored beans increases as you increase the number of black jelly beans there. Your intuition also tells you that the concentration of black jelly beans decreases if you increase the size of a pile of beans that contains always 15 black ones. Intuition means a judgement not based on conscious reasoning. Sometimes your intuition is wrong and sometimes it is right. Either way, a good rule for the scientist is: Never ignore your intuition. If you have an intuitive feeling about something, you should investigate it. A scientist's intuition will often lead him to important discoveries or, just as importantly, to errors he may not otherwise have noticed. Of course, you never base a conclusion finally on intuition, but a strong hunch is always worth investigating to see whether logic and experiment bear out the conclusion your mind leaped to. The difference between a scientist and a gambler is that the scientist. applies logic and experiment to his hunches to see whether intuition is supported by reason.

A simple illustration is to see whether your intuition about the jelly beans above agrees with the definition. If you have a box of a certain size filled with jelly beans of different colors, your intuition tells you that the more black beans you put in the box, the greater is their concentration. Now what does does the definition say? Concentration is the quantity you get when you divide the number of black beans by the volume of the box. If you have a box of constant volume, then the concentration is found by dividing the number of black beans always by the same number. That is, concentration is then a fraction whose numerator may change but whose denominator is constant. You know that under these circumstances, the larger the numerator, the larger the value of the fraction (5/17 is bigger than 4/17, for instance, or 2. 3/5 is bigger than 2. 1/5). Thus the more black jelly beans (i. e., the larger the numerator) when the box is fixed in size (i. e., the denominator is constant), the greater is the concentration (i. e., the value of the fraction). Your intuition was okay in the first case.

Your intuition also tells you that for a fixed number of black jelly beans, the larger the box into which they are mixed, the smaller the concentration. This time the numerator is constant and the denominator changes. Does the value of the fraction decrease as you increase the denominator? (Is 3/17 smaller than 3/16? Does the quotient get smaller if you divide a certain number by successively larger numbers?) Is your intuition right again?

Would intuition tell you to expect the density of a sugar solution to depend upon its concentration? Let's do Experiment 20.

Points to Discuss in Class

Did you happen to notice that the units of concentration and the units of density are the same? Don't let this bother you; it happens now and then in physical science that two entirely different qualities are measured in quantities having the same units. Density and concentration may be different qualities, but they may have the same units. The point is that one must not confuse the quality with the units in which the quality is measured. This point was mentioned before on page 53. The head of the laboratory in a paint company instructs one of his technicians, "Measure the pounds per gallon of this paint." How can the technician know what his boss wants? What two things are likely to be the datum the boss is seeking?

Do you see how, though an idea may be simple, it may be a rather involved process to reduce the idea to numerical measure? You think to yourself: "Density and concentration are simple ideas. To get the density of this solution, all I need is the weight and volume of a sample of it. To get the concentration, all I need in addition is the weight of sugar in that volume. Easy, let's go measure them." And then you see that measuring them turns out not to be so direct and easy a business after all. Sometimes physical measurements, though simple in meaning, have to be measured by very indirect and elaborate methods.

What does your graph look like? Is density a function of concentration? Can you find the density of any concentration you are given (within the range of the graph), even though it is one that no one in your class happened actually to measure? Is density a monotonic function of concentration? An increasing function? Is the density proportional to the concentration? The graph of this function is a straight line; or better, is so nearly a straight line as to allow being considered so for most purposes. If you have not already done so, use a ruler to draw what looks like the best straight line through the points. Remember that experimental error will inevitably find some of the points a little off. Try to draw the line so that you leave as many off-points on one side of it as on the other.

7. Linear Functions

The density of a sugar solution is an increasing function of the concentration, for the graph slopes always upward to the right. The steepness of the slope is constant, because the curve is a straight line. The density is not proportional to the concentration, however, because the straight line does not go through the origin. Any function whose graph is a straight line is called a linear function. A linear function is always monotonic, because the slope never changes. Is a monotonic function always linear? A linear function may be either increasing or decreasing, depending upon whether the straight line slopes upward or downward. A proportional function is always a linear function, because a proportional function is merely a special case of linear function in which the graph happens to go through the origin. Is a linear function always a proportional function?

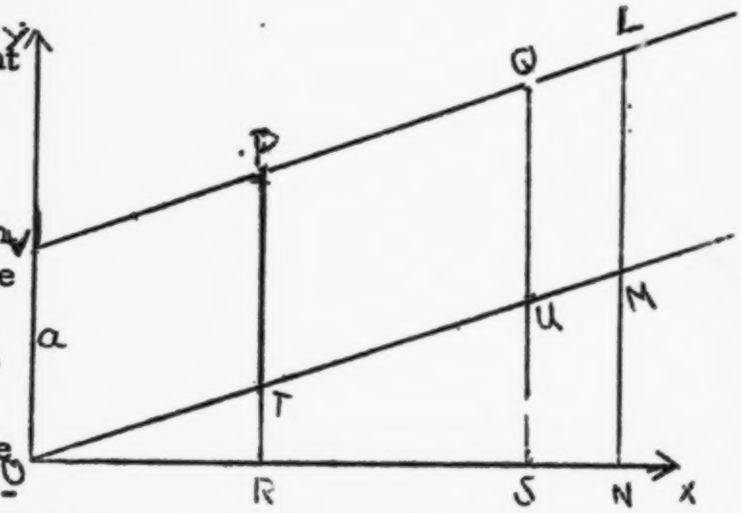
Let's go symbolic again! Suppose that Y, a dependent variable, is proportional to X, the independent variable. We learned before, you remember, that: (1) the graph of Y vs. X is therefore a straight line; and (2), there is some constant, k, such that $Y = k \times X$ for all the possible Y's and X's.

It is time now for us to get used to a certain convention regarding the "times sign", X. When symbols are used to represent quantities, we have so far written their product using the "times sign." When we wished to represent the the quantity "A times B", we would write it as "A x B." Most people, however, simply omit the times sign when the quantities are represented by symbols. This we shall do. Hereafter, when we wish to write the product of "A times B", we will simply write "AB." Then when you see two symbols written together this way, remember that it always means the product of the two. You will quickly get used to this convention.

Two things you have to be careful about, however. One is that you use the convention only when at least one of the factors is not a number. You may write "A times B" as "AB"; you may write "2 times B" as "2B"; but you must continue to write "2 times 2" as "2 x 2". Do you see why? The other thing to be careful about is that you never choose more than one letter as a symbol for some quantity. For instance if you chose to represent the extension of a spring as "EX", no one could tell whether you meant "extension" or the product of the two quantities, E x X. Sometimes two letters are used to symbolize a single quantity; if so, the symbol usually carries a bar over it to show that they are tied together: like $\overline{EX}$.

It is a characteristic of a proportional function, you remember, that it can always be represented by the very simple analytical expression, $y = bx$, where y is the dependent variable, x is the independent variable, and b is the name of some constant whose value depends on the slope. The close interconnection between the equation, $y = bx$, and a straight line through the origin leads to the usage of speech: "The equation of a straight line through the origin is $y = bx$." Since the equationof a straight line through the origin is so simple, one wonders whether it is possible to find the equation of a straight line that does not go through the origin. Such an equation would then be an analytical representation for any linear function just as "$y = bx$" is an analytical representation for any proportional function.

This can indeed be done. Imagine any straight line not through the origin, like the upper line in the graph to the right. Think very carefully what this graph means. Consider how the point P might have been plotted. You would be given a value for y and a value for x. Regardless of what kind of quantities y and x represent, you always think of them as distances when you make a graph. The point P must then have been plotted with x equal to the distance $\overline{OR}$, and y equal to the distance $\overline{PR}$. Similarly Q represents the combination x = $\overline{OS}$, y = $\overline{QS}$. We write bars over the two letters (like $\overline{PR}$) to indicate that $\overline{PR}$ is one symbol for a certain distance -- not the product of two quantities, P times R. Now draw a line through the origin parallel to the first line, PQ. Suppose this new line intersects the two lines PR and QS at T and U as labeled in the drawing.



This new line we already know can be represented by the equation, Y = bX (where we shift to capital letters to avoid confusion with the small letters we are using for the other line). What does this mean? Well it means that any point (like T) has an X and Y such that X = $\overline{OR}$, V = $\overline{TR}$. (Remember that any pair of letters with a bar over it represents a single distance. In other words, it is a quantity.) Moreover, these two quantities X and Y are related by the equation Y = bX. That is,

$$\overline{TR} = b \times \overline{OR} \tag{1}$$

Can you write a similar equation derived from the point, U? It would be

$$\overline{US} = b \times \overline{OS} \tag{2}$$

You should see very clearly that these last two equations are nothing more than two cases of Y = b x X, an equation that holds for every point on the line OTU.

Now since the two lines VPQ and OTU are parallel, the distances $\overline{OV}$, $\overline{PT}$, and $\overline{QU}$ are all equal. This is a property of parallel lines that you are probably familiar with and that we will make use of here without proving it. You will prove it when you study geometry.

The two distances $\overline{PR}$ and $\overline{QS}$ can obviously be written as the sums of their components:

$$\overline{PR} = \overline{PT} + \overline{TR}$$

$$\overline{QS} = \overline{QV} + \overline{US}$$

- - -

But in the paragraph just before this, we saw that PT and QU are equal -- that is, they are different names for the same quantity. Suppose we give this quantity still another name, a, which is written in on the figure for you as a label for OV (which happens to be still another name for it!). Now we can replace PT and QU in the last two equations above by their other name, a, and write

$$\overline{PR} = a + \overline{TR}$$

$$\overline{QS} = a + \overline{US}$$

The quantities TR and US also have other names; they are given in equations (1) and (2). If we replace TR and US in the last two equations by these other names, we have

$$\overline{PR} = a + b \times \overline{OR} \qquad (3)$$

$$\overline{QS} = a + b \times \overline{OS}. \qquad (4)$$

If you look at these last two equations, you can't fail to see their similarity: they both involve a and b in the same way. But they have a similarity even more striking. If you look back at the drawing, you will see that $\overline{PR}$ and $\overline{QS}$ are simply the y's of the two points P and Q; and $\overline{OR}$ and $\overline{OS}$ are simply their x's. In other words, equations (3) and (4) say that for the two points P and Q on the upper line, it is true that

$$y = a + bx \qquad (5)$$

Now this conclusion would be true for any point at all on the upper line, for there was nothing special about the points P and Q that would make the conclusion hold for just those points. In fact, why don't you try yourself to go through the whole argument using the point L as labeled on the drawing? Furthermore, there was nothing special about the line VPQ: it could be any straight line at all not going through the origin. Therefore equation (5) is the equation of any straight line not through the origin.

It is important that you understand the meaning of equation (5). If you draw any straight line on a graph, every point on the line has some certain y and x. The y of any point on the line depends upon which point you are talking about. You can specify any point you wish to call attention to by naming its x. That is, once you name an x, there is only one point on the line that has that x. Therefore you see that specifying an x specifies a point on the line, and specifying a point on the line specifies its y. Thus, as long as you are talking about points on this line, whenever x is specified, then automatically y is fixed. As long as you are talking about points on this line, in other words, y is a function of x. If you were talking about some other line, specifying the same x would in general give you a different y, as you will readily see if you draw two different straight lines on a graph. Thus each different line you draw will give you different y's for the same x; each different line, in other words, represents a different function of x.

We have seen that the y and x of every point on a line are connected by the analytical representation

$$y = a + bx. \qquad (5)$$

It is customary to say that "$y = a + bx$" is the equation of a line. Each different line, of course, has a different combination of a and b; but once you have chosen some particular line, the a and b for that line are constants. The reverse is also true: once you have chosen a and b, there is only one straight line you can possibly get. You should now take a piece of graph paper an and try plotting the graph of equation (5) for some particular choice of a and b. Suppose you choose $a = 2$ and $b = 3$. Make yourself a little table like this:

x	0	1							
y	2	5							

Choose a series of any values you wish for x; calculate the corresponding y from the equation, $y = 2 + 3x$. Write the chosen x's on the first line of the table and the calculated y's on the second line. After you have 6 or 8 pairs of x-and-y, plot them on the graph and see that they form a straight line. It would be a good idea then to choose another combination of a and b and repeat the whole operation to see that this time, too, youget a straight line, but a different one, of course.

Keep in mind now that every point on a given straight line has an x and y such that $y = a + bx$, where a and b are some fixed numbers. What is the value of y when $x = 0$? You see immediately that, regardless of what a and b are, y has the value a when $x = 0$. Thus every straight line, whose equation is $y = a + bx$, crosses the vertical axis (where $x = 0$) at a distance a from the origin. In other words, if you see a straight line plotted on a graph, you know first that that line has the equation $y = a + bx$, where a and b are some fixed numbers; and you can tell at sight what the value of a is for that line. It is simply the value of y when $x = 0$, or the point where the line crosses the y- axis. It is less easy to tell the value of b at sight.

In the particular case where the line passes through the origin, the distance a is of course zero. Hence for a proportional function, $y = 0 + bx$, or $y = bx$, as we learned before. Notice then that a proportional function is a special case of a linear function in which the constant term (a) is zero or in which the line crosses the y-axis zero-distance from the origin.

Now finally let's get back to your sugar solutions. Look at the graph you made from the data of Experiment 20. You found that the graph is a straight line that does not pass through the origin. The graph is a representation of the functional relationship between density (d) of a sugar solution and its concentration (c). We asked whether we could also find an analytical representation of the same functional relationship. You now know that such an analytical representation has to be of the form, $d = A + Bc$, where A and B are two constants. The trouble is that we do not know the actual values of these constants. Can we find them? Yes, we can.

In the first place, you know immediately that A is the value of d when c = 0. That is, A is simply the density of a sugar solution whose concentration is zero, or the density of plain water, 0.997 g/cc. Thus we can write immediately that

$$d + 0.997 + Bc \qquad (6)$$

when the density is given in g/cc.

Now this equation says that "d" and "0.997 + Bc" are merely different names for the same quantity. If I subtract 0.997 from this quantity, I will get the same result whether I subtract 0.997 from "d" or from "0.997 + Bc". That is,

$$d - 0.997 = 0.997 + Bc - 0.997.$$

Notice that on the right, we are taking the number Bc and adding 0.997 to it and then subtracting 0.997 away again. This leaves us of course with just Bc. Hence

$$Bc = d - 0.997,$$

and this equation holds for all c and d. Now solve this equation for B and you get

$$B = \frac{d - 0.997}{c}.$$

In this last equation, we do not know B, but we do know lots of combinations of of c and d. If you select (from the graph or from the table) a pair of values of c and d that go together, you can place these values in the right-hand side of the last equation, and work out numerically the right-hand side. If the c-and-d combinations all fall on the same straight line, you will get the same value of B no matter which c-and-d combination you use.

Do you see the reason for the last statement? It isn't magic or dumb luck! The point is this: all the c-and-d combinations fall on one straight line. Therefore, as we proved, they have to obey an equation of the form "d = A + Bc." In our particular case, every c-and-d combination must obey the equation, "d = 0.997 + Bc", where B is the same constant for every combination. Now we showed by logic that: If it is true that d = 0.997 + Bc, where B is a constant regardless of what c and d you are talking about, then it is true that B = (d-0.997)/c where B is a constant regardless of what c and d you are talking about. Okay?

Everyone in the class should now calculate B from his own c-and-d combination, which is the second line of Table II in Experiment 20. Everyone will get nearly the same value of B; not exactly the same because of experimental error. Take the average of all for the best value of B. Now you can write equation (6) with the numerical value of B put in. You will get an equation very close to

$$d = 0.997 + 0.378\,c.$$

Observe that this is a functional relationship between density and concentration for sugar solutions. You can now predict the density of a sugar solution if you are given the concentration

8. Once Again, Lightly

One must be careful in precise speech to be sure that such words as "size," "amount," "big," etc., are used so that their meanings are understood: or else avoid using them. For example, "length," "volume," and "weight" are all different but possible meanings your listener may attribute to your use of the word "amount."

Density of a material is the ratio of its weight to its volume, and is a constant property of that material, regardless of which piece of it your are talking about. Density can also be regarded as the proportionality constant in the statement "The weight of a piece of material is proportional to its volume." A determination of density can often be used to establish the identity of the material of which a thing is made.

The concept of density applies to liquids, too. The density of a solution is a function of its composition. For a solution of known components, its composition can often be determined when its density is known.

Concentration is a numerical expression of "crowdedness" and is defined as the ratio of the amount of material to the space occupied by that material.

Any function whose graph is a straight line is called a linear function, and is always of the form $y = a + bx$. In this equation, a is the distance from the origin where the straight line meets the y-axis. When the function is proportional, this distance is zero, and the equation becomes simply $y = bx$.

Unit V

Motion in a Straight Line

1. Position and Distance

This unit deals with certain aspects of the motion of moving bodies. As you know, motion is the business of going from one place to another; that is, motion is a change in position of a body. Notice that one cannot observe the motion of a body unless he can observe its position at some moment and again at some later moment. Thus in order to talk about motion, we have to be able to talk about position; especially, we have to be able to tell the person we are talking to just where a body is.

Notice that telling someone where something is, is really the same thing as giving an address. Here are some examples:

Five blocks north and three blocks east of the postoffice
Twelve paces south and twenty paces wast of the elm tree
35° north latitude and 131° west longitude
Ten inches from the corner of the table along the front edge.

You might try making up some "addresses" like this yourself. Can you, for instance, tell someone how to find the pole-star in the sky? Notice that you cannot tell anyone where the pole-star is -- or where anything else is, for that matter -- without telling how far it is from something else. Look at the examples above; they all fix an address by using some fixed reference point: the post office, the elm tree, the corner of the table. What is the fixed reference point in the third example? Most people locate the pole-star by using two stars in the big dipper as reference marks. To repeat: you can locate a body only if you tell how far it is from something else. "Far" and "from" -- distance and reference mark. You can see how the idea of motion is tied up willy-nilly with the ideas of distance and position.

Now suppose you were way out in space by yourself -- so far away from anything else that you could be regarded as completely alone. Question: Are you moving or standing still? You might find this a little shocking, but the modern scientist would say that this question has no meaning! For the only way you could speak meaningfully of your motion would be to speak of your position at one moment and your position at a later moment. But you are alone; there is no reference mark available to describe your position, and therefore no way to tell whether you are moving. The modern scientist would say, since it is not possible to learn whether you are moving, that the idea of motion is without meaning to you. You might well say "But even though I can't tell whether I am moving, this doesn't mean that I have no motion. It's only that I can't tell. If I'm in a train with my eyes closed, I may not be able to tell that I'm moving, but this doesn't mean that I'm not." You have a good point, and it has been argued by scientists and philosophers for many years. The point is that if the concept of position is without meaning to you, so is the concept of motion.

Perhaps you could understand this more clearly if you imagined yourself born in a spacecraft, floating in a sea of total emptiness all your life and alone. Suddenly a voice from nowhere asks:

"Where are you?"

"I don't understand the question," you reply. "I'm here. There isn't anywhere else."

"Well, where are you going?"

"Again, I don't understand what you mean by 'going.' There is no place to go to; I can't tell one place from another. I'm here and I cannot be going anywhere. I'm not moving, for I don't even know what you mean by 'moving'."

Do you see that the "voice" couldn't even explain to you what the word "move" means? The words "move" and "motion" literally have no meaning to you.

But let's get back to where we can describe the position of a body and therefore tell where it is and whether it is moving. At first we will speak only of motion along a straight line. This means that no matter when we observe the body, it will always be somewhere on this line. We can then conveniently describe the position of the body by choosing some point on the line as the reference point and stating how far the body is from that point.

Notice, however, that there is an uncertainty here. If I say the body is 21 cm from the reference point, you will not know whether I mean 21 cm to the left or to the right of the origin (another name for the reference point). Let us then agree to the following convention. We will call one side of the origin "plus" and the other "minus". If the body lies on the minus side, we will call its position "-21cm"; if the body lies on the plus side, we will call its position "+21 cm". What should we call the position if the body lies right on the origin?

Next, we must settle which side is to be plus and which minus. The choice is only a matter of taste, of course, and mostly it doesn't matter which we choose as long as we agree on it. We shall use the following convention unless you are told otherwise: When the body is moving along the line, we will say it is moving away from the negative side and toward the positive side. In other words, if you stand so that the body is moving to your right, then the minus side is on your left and the plus side on your right. Okay?

Still another way to look at it is to notice that the body is always moving toward larger numbers. If the body is now at +10, it will later get to +15; if it is now at +2, it will later get to +6; if it is now at 0, it will later get to +3; if it is now at -2, it will later get to +2. Also, if it is now at -10, it will later get to -5. Notice that -5 is a larger number than -10. You will have to get used to the idea that -A is bigger than -B whenever A is smaller than B. You use the same idea when you say that -15 degrees is warmer than -30 degrees; or that the second floor below ground is higher than the fourth below; or that 200 BC is later than 300 BC. It's an easy idea to grasp, so don't let it get you mixed up.

Now we can easily find the distance between two points. The distance between two points is simply the difference between their positions. (Actually, you used this fact in all your spring experiments.) Again there might be an ambiguity. If you are given two positions, A and B, is the distance between them A-B or B-A? Here again we will have to agree on one or the other. Since the word "distance" as we are using it here means "distance the body has traveled," we want to subtract in the way that shows how far the body traveled; or, what is the same thing, we want to show how much its position has changed. Now when we speak numerically of a change, we always in common speech mean "second minus first." How much did you grow in height in this year? To answer you subtract your height last year from your height this year: second minus first. How long did it take you to paint the fence? To answer you subtract the starting time from the finishing time: second minus first. How much did the temperature change from noon to midnight? You subtract the noon temperature from the midnight temperature: second from first.

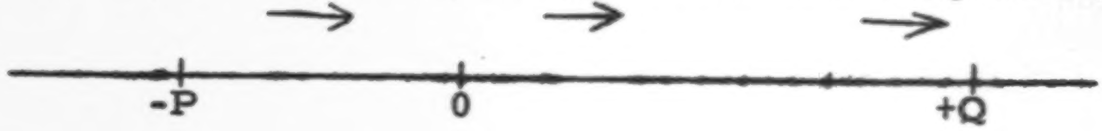
So in our case also. "Distance traveled" always means "final position minus initial position." Since the body, in accordance with our convention, always travels toward higher-number positions, we then will be subtracting a smaller number from a larger. Examples:

How far does a body travel if it starts at +10 cm and ends at +25 cm? Answer: (+25) - (+10) = 15 cm.

How far does a body travel if it starts at -10 cm and ends at +25 cm? The answer would be given by "(+25) - (-10)", but how do you work this out? You must remember that the only things we know how to subtract are ordinary numbers. Numbers like 16, 1.97, 2/3, $\sqrt{2}$, +25, and 0 are "ordinary" numbers, but what is this thing we are calling "-10"? To be more precise, we will call those numbers that lie on the plus side of the origin, "positive" numbers (instead of "ordinary" numbers); these new things that lie on the minus side we will call "negative" numbers. To repeat then: the only kind of numbers you know how to subtract are positive numbers. What does it mean to subtract a negative number? Subtraction of negative numbers has never been defined for us, and it therefore does not yet have a meaning. We can give it any meaning we want to. The first question then is not "What does 'subtracting a negative number' mean?"; for it doesn't mean anything yet. The first question is rather "What do we want 'subtracting a negative number' to mean?"

To decide what we want it to mean, we have only to look at how the whole idea of subtracting a negative number arose. It came up because we defined "distance a body travels" as "B-A", where B is its final position and A its initial position. Since it is possible that A be negative, we immediately run into the possibility of having to subtract a negative number. Whatever it means to "subtract a negative number," then, we want the result of "B-A" to mean the distance a body travels in moving from position A to position B, even when A is negative. So let's consider carefully the travel from a position, -P, to a position, +Q.

In making this trip, the body might be thought of as moving from the position, -P, to the position, zero, and then on to the position, +Q, like this:



In the second part of the trip (from zero to +Q) the distance traveled is (Q-0) or simply Q. To get the total distance traveled, we must then add something to Q. The total distance we have defined as "Q - (-P)", however, so that "subtracting (-P) from Q" has to mean adding something to Q. What must you add? Now the length of the first part of the trip from -P to 0 is actually just +P. Why? Because that's the way -P was defined to begin with: If you lay off a certain length to the right of the origin, we will call it +P and if you lay off the same length to the left of the origin, we agreed to call it "-P". Thus the total trip is Q + P. Thus we feel that we would like to define "Q - (-P)" in such a way that

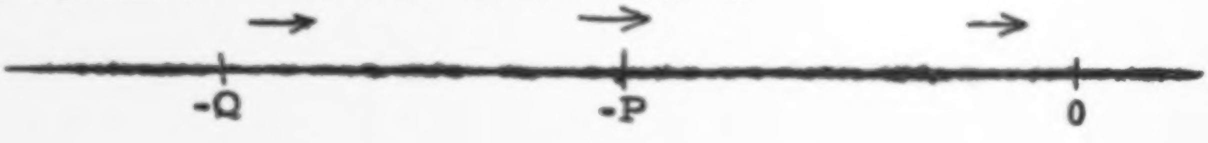
$$Q - (-P) = Q + P \qquad (1)$$

It is worth pointing out to you again that the two preceding paragraphs are not a proof that Q - (-P) is the same thing as Q + P. The intent of these two paragraphs is rather to show you that equation (1) is a reasonable definition of what ismeant by "subtracting a negative number." We accordingly take equation (1) as a definition of what the previously undefined operation of "subtracting a negative number" will henceforth mean. It is important that you see that we could have defined it in anyway we wanted; but the way that we finally choose to define it has the important property that then the distance from -Q to +P gives what we intuitively feel "Q - (-P)" ought to mean. Now how far does a body travel if it starts at -10 and ends up at +25? Work it out yourself.

Then finally we ask: How far does a body travel if it starts at -Q and ends at -P? The definition of distance says that this distance is "(-P) - (-Q)". We have already decided that "subtraction of (-Q)" means "addition of Q." Hence "(-P) - (-Q)" means "(-P) + Q". Since addition is commutative (that is to say, we want it to be commutative if we can get it so), we can rearrange "(-P) +Q" to "Q + (-P)". Then we would like it to be true that

$$\text{distance from } -Q \text{ to } -P = (-P) - (-Q) = Q + (-P). \qquad (2)$$

In a picture, the situation looks like this:



Now you can see that the distance from -Q to zero is more than the distance from -Q to -P. Again, you can seethat

(the distance from -Q to zero) exceeds (the distance from -Q to -P) by a (distance equal to that from -P to zero.)

But the distance from -Q to 0 is Q (Remember? That's the way we defined -Q to begin with!), and the distance from -P to zero is P. Therefore the indented sentence above can be translated by replacing the contents of the first parentheses by "Q"; the contents of the second parentheses by "Q + (-P)" (you get this permission from equation (2)); and the contents of the third parentheses by "P". That is,

Q exceeds Q + (-P) by P.

Next, we must realize what "exceeds" means. To say that "U exceeds V by W' means that "U is W more than V" or "U = W + V". Thus we can write

$$Q = P + Q + (-P).$$

Now subtract P from both sides of this equation. Then

$$Q - P = P + Q + (-P) - P.$$

Now right away you see that we are both adding and subtracting P on the right. Canceling them out gives

$$Q - P = Q + (-P).$$

Turning this equation around will make the point a little more clear:

$$Q + (-P) = Q - P.$$

In words, adding -P to something is the same as subtracting P.

We can write equations (1) and (3) together like this to exhibit them more compactly:

$$Q - (-P) = Q + P$$

$$Q + (-P) = Q - P$$

Notice the symmetry: the first equation says that subtracting -P is the same as adding P; the second equation says that adding -P is the same as subtracting P. That's easy, isn't it? Whenever you want to add or subtract a negative number, you simply drop the minus sign and do the opposite.

Once again: these rules for dealing with negative numbers were not derived or proved. They are definitions of what we shall henceforth mean by adding and subtracting negative numbers -- something that had not heretofore been defined. All the discussion was only to show the definitions to be reasonable and consistent with the ones for positive numbers.

2. Velocity

A body that is moving is by definition a body that changes its position. You have many times observed that two different bodies can change their positions at different rates. A rabbit, for example, can change its position from here to there more quickly than a turtle can. You know already -- and we have several times before used this information -- that the speed of a body is defined as the (distance the body travels) divided by (the time it takes to travel that distance). Suppose, as a body moves along, that you measure the time it takes to cover many different intervals. Suppose further that you find the ratio, distance/time, to be constant for all the intervals. Then we say that the body has a constant speed. For the present, we confine our attention to bodies moving at constant speed.

The definition of speed as "distance/time" is familiar to you, but have you ever wondered why it is defined that way rather than, say, as "time/distance"? The reason is closely related to the reason why concentration is defined as "amount/space", and you should take off a few moments to think about it. The meaning of the word speed, as you grew up using the word and hearing it used, is such that the greater speed is to be assigned to the body that travels a given distance in the shorter time. If you have two bodies traveling the same distance in different times, then, their speeds will be given by two fractions whose numerators (distance) are the same but whose denominators (time) are different. Which of the two fractions has the greater value -- the one with the smaller or the one with the larger denominator? Does this agree with what you want speed to mean? If you have two bodies traveling over different distances in the same time, you want the one that travels the greater distance to have the greater speed. If they travel different distances in the same time their speeds will be given by two fractions having the same denominator (time) but different numerators (distance). Which of the two fractions has the greater value -- the one with the larger or the smaller numerator? Does this agree with what you want speed to mean? So you see that someone's suddenly telling you that speed means "distance/time" is not violating the conception of the word that you already have. The new definition merely makes precise and numerical what you already had in mind.

The two words, speed and velocity, have slightly different meanings. The difference between them will concern us later; but as long as the motion is along one straight line, their meanings are identical.

It is now time to do Experiment 21.

Points to Discuss in Class

Are the three curves straight lines? Use a ruler to draw the best straight line you can for each plot. Do you notice any special relationship among the three lines?

Since the curves are straight lines, they represent linear functions. This means that the position attained by a body moving at uniform velocity is a linear function of the time. If we represent position attained by p and time by t, in other words, it must then be true that

$$P = A + Bt, \qquad (4)$$

where A and B are constants for any one curve. They (A and B) may, of course, be different constants for the different curves; all we know is that for any one straight line, there will exist some A and B which have always the same value for that line. This means that for any one travel of the body, there will exist some A and B such that you can always calculate p from equation (4) when t is given to you.

Can you find the values of A and B for your curves? You have already learned (page 110) that the value of A for any linear curve whose equation is equation (4) is the value of p when t equals zero. But the value of p when $t = 0$ is the value of p at the point where the curve crosses the vertical axis (the p-axis). You can then tell the value of A for any of your straight lines merely by looking at the graph and reading the value of p at the place where the curve crosses the p-axis.

But more than that, do you have a feeling for the meaning of A? When you say "$t = 0$", you are referring to the instant at which you started counting time. What is the position (p) that the body has attained since you started counting time? Well, this position is given, for any t, by equation (4). In particular, what position has the body attained since the starting time up to the time when $t = 0$? You can see that the time "$t = 0$" is the starting time; therefore the body is at this moment just on the verge of moving away from where it was at the starting time, but of course has not yet left there. At time $t = 0$, then, the position the body has attained is the same as its starting position. Now notice the elegant consistency among these three things:

(1) Your reason tells you that the position of the body at time $t = 0$ is the same as its starting position.

(2) Equation (4) tells you that the position of the body at time $t = 0$ is A.

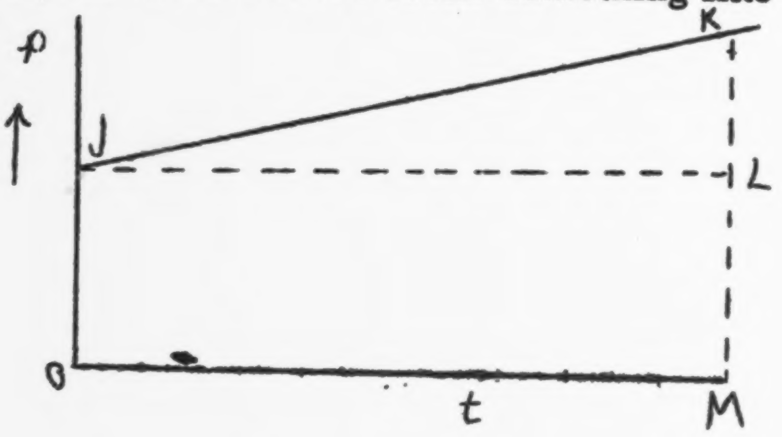
(3) Your graph tells you that the position of the body at time $t = 0$ is given by the point where the curve crosses the p-axis.

Taken all together, then: your reason, the analytical representation given by equation (4), and the graphical representation given by your graph tell you that:

the starting position of the body
and the value of A
and the point where your graph crosses the p-axis
are all the same thing!

Do you see how neatly these all fit together? Do you see how the graph makes vividly visual both what your common sense tells you of the motion of the body, and what equation (4) allows you to calculate about the motion of the body?

One of your three curves looks something like the JK in the following diagram:



You realize now that the line-segment ("line-segment" means piece of a line) $\overline{OJ}$ is simply the value of A in the equation $p = A + Bt$. This curve represents a monotonic increasing function. Notice again that your reason and your observation in doing the experiment show that the change in position of the body as it falls down the tube is indeed monotonic, for the body falls steadily downward (in the direction of higher numbers on the measuring tape) without ever falling up. The graph pictorializes this observation. Try not to be confused by the arrow at the left on the diagram: it points up as the direction of increasing p, whereas the body actually fell down as p increases. Increasing p means motion downward in this experiment.

Suppose now we ask the question: What would be the position of the body at any time if it starts at J and stays thereforever? The answer of course is that the body is always at the same position; that is the curve is always the same distance, $\overline{OJ}$, from the t-axis. This line is drawn dotted on the diagram. Do you see why this dotted horizontal line represents the "motion" of a body that starts at a distance OJ from the origin and never moves away? If it never moves, then its distance from the origin, $\overline{OJ}$, stays always the same. The distance from the origin at a later time (say at time M) must be the same as its distance at the start. The distance at later time M is $\overline{ML}$, and its distance at the start is $\overline{OJ}$. The only way this distance would always be the same -- $\overline{OJ} = \overline{ML}$ = the distance at any other t at all -- would be if the line JL is parallel to the t-axis. So you see that a body that does not move (which means a body moving with zero velocity) can have its motion represented on a p-t graph by a horizontal line whose distance from the t-axis is always the same.

Now consider the case you actually investigated, where the body does move; where as time goes on, the distance from the horizontal axis does not stay the same but continuously increases. Such a motion must be represented by a line that slopes upward to the right; for as you move to the right on the graph (moving to the right means "as time goes on", doesn't it?), the body increases its distance from the reference mark -- that is, moves farther and farther away from the reference mark. Therefore as you move to the right on the graph, you must represent the position of the body by points higher and higher on the graph. You can probably see also that: if the body is moving fast, then the graph must rise steeply, because the position of the body moves faster away from the reference mark; and if you move slowly, the graph rises only gently, because the position of the body moves only slowly away from the reference mark.

For instance, if the body is not moving at all, then by the time M (look at the diagram) the body will not have moved any farther away from the reference mark than it was at the beginning, and the position of the body will be given by ML, which equals OJ. If, however, it is moving very rapidly, then by the time M it will have moved well beyond its original position and KL will be large. If it is moving only slowly, then by the time M, it will not have moved very far beyond its original position and KL will be small. Perhaps you can see that for a body of high velocity, the line JK will be steep because it must rise rapidly whereas for a body of lower velocity the line JK will be less steep. If in fact the body is not moving at all, then the line JK would have no steepness at all! It would be perfectly flat like JL. The faster the body moves, the steeper will be the line JK.

Now do Experiment 22, which will help you understand the relationship between the steepness of the graph and the velocity of the moving body.

Points to Discuss in Class

All the curves in Experiment 22 are straight lines. Are they all equally steep? Could you have predicted whether the faster fall would have had the steeper or the more gentle slope? Can you tell merely by looking at the graphs which curve goes with the highest velocity? Recall the discussion on page 97 regarding the steepnesses of several curves you previously drew. For which of two moving bodies does position increase more rapidly -- the slow body or the fast one? Which curve rises more rapidly -- that for the slow or the fast one? Which curve is steeper -- that for the slow or the fast one?

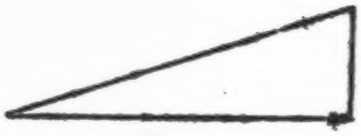
You should now have a feeling for the fact that the steepness of the curve -- position vs. time -- is somehow related to the body's velocity. The situation is entirely analogous to the curves of weight vs. volume that you obtained in Experiment 16, where the steeper curve went with the greater density. We want now to examine certain numerical aspects involved in the idea of "steepness."

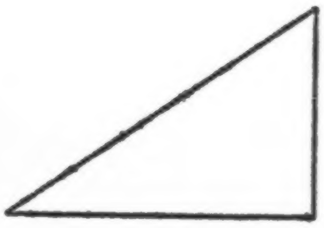
3. Slope of a Straight Line

"Can you tell which of two ramps is steeper just by looking at them?", Tom asked Jerry.

"Sure, that's easy," answered Jerry. "Try me."

Tom showed Jerry a piece of paper on which he had drawn two straight ramps. The drawings looked like this:





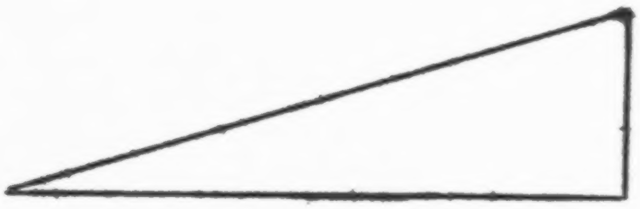
"Okay, Jerry, which one is steeper?" Tom asked his brother.

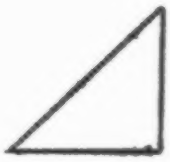
"Why the one on the right, of course," Jerry replied.

"Why do you say the one on the right?"

"Because it goes up higher than the one on the left." Jerry went on to explain, "The steeper the ramp the higher up it goes."

Tom wasn't quite so sure. "Now wait a minute," he cautioned. "Look here. I'll draw two other ramps, and then you tell me which is steeper." Tom then made two new drawings that looked like this:





"Now, which one is steeper?" he asked.

"The one on the right again," answered Jerry.

"But the one on the left goes higher," Tom reminded his brother. "According to what you just told me, you ought to call the left one steeper."

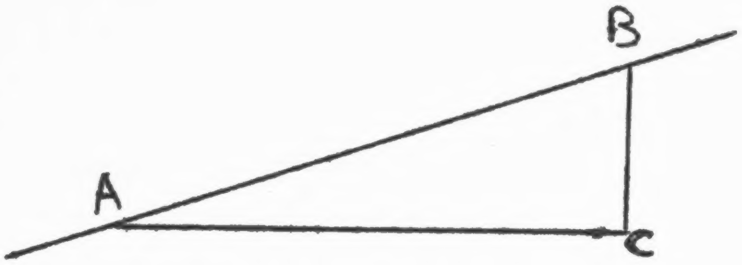
"Yes," admitted Jerry, "I guess I spoke too fast. The left one goes higher, yet I can see that the one on the right is steeper. There must be something more involved in 'steepness' than just how high the ramp goes. I'm not so sure any more."

Do you think you can help Jerry to formulate his intuitive idea of steepness into something numerical and definite?

The idea involved is analogous to those involved in the distinction between weight and density, or between amount and concentration. In the present case we are trying to avoid the confusion between "height of a ramp," and "steepness of a ramp." They are closely related, you realize, but are not the same.

Suppose you were an ant in the middle of the ramp somewhere, yet with all your human sensibilities. You can see neither end of the ramp and have no idea how far it goes up or down. Could you still tell whether one ramp was steeper than another? Of course you could, so right away you know that the height of the ramp is not at all what determines its steepness. When you determine the density of a material you take a unit volume of it and determine its weight; when you determine the concentration of a solution, you take a unit volume of it and determine the amount of material dissolved in it. What counts in density is not merely the weight, but the weight per unit volume. What counts in steepness is not the height, but how much the height increases per unit of horizontal distance. This, if he had the necessary instruments, an ant could determine.

Notice that we said above that steepness could be thought of as the amount the height increases per unit of horizontal travel. Consider the ramp below.

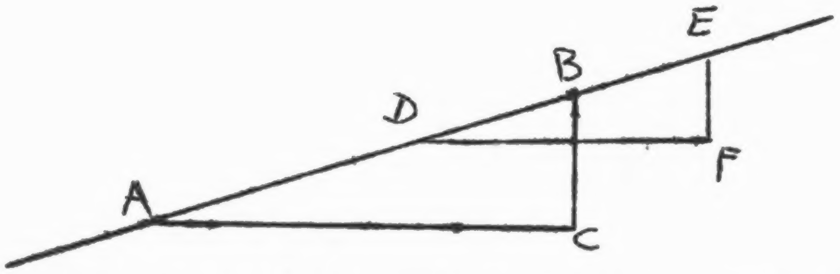


We could select any two points we wished on the ramp, say A and B. We could then lay out a horizontal line from A, like AC, and a vertical line from B, like BC. Then $\overline{BC}$ is the amount by which your height increases as you walk from A to B, and $\overline{AC}$ is the amount of horizontal travel. To find the "height increase per unit of horizontal travel", you would then divide $\overline{BC}$ by $\overline{AC}$ (they are both numerical quantities, remember!).

There are two questions you probably now are asking. One of these is: "But can I not express the steepness in other ways that are just as good? For instance, why not say the steepness is simply the value of the angle at A? Or why can't I say that steepness is "height increase per unit of travel along the ramp" -- rather than per unit of travel horizontally?" The answer is that you can. This a a cat that can be skinned in several ways. For our purposes, as you will soon see, it will be more useful to use the first suggestion above, however, -- the "height increase per unit of horizontal travel." To avoid ambiguity in the word "steepness", then, we call this particular measure of steepness, slope, and we then have the definition:

$$\text{slope} = \frac{\text{Increase in Height}}{\text{Horizontal Distance}}$$

The other question you were about to ask is this: The two points, A and B, *on the* ramp were just chosen at random, and then the numerical value of $\overline{BC}/\overline{AC}$ was computed and called the slope of the line. Suppose someone else had measured the slope of the ramp. He probably would not have chosen the same two points A and B. In the drawing below, for instance, perhaps



the other person chooses the two *points,* D *and E.* Then *he* calls the slope $\overline{EF}/\overline{DF}$. Now what *is* the slope: $\overline{BC}/\overline{AC}$ or $\overline{EF}/\overline{DF}$ or perhaps something else that still a third person might measure and compute?

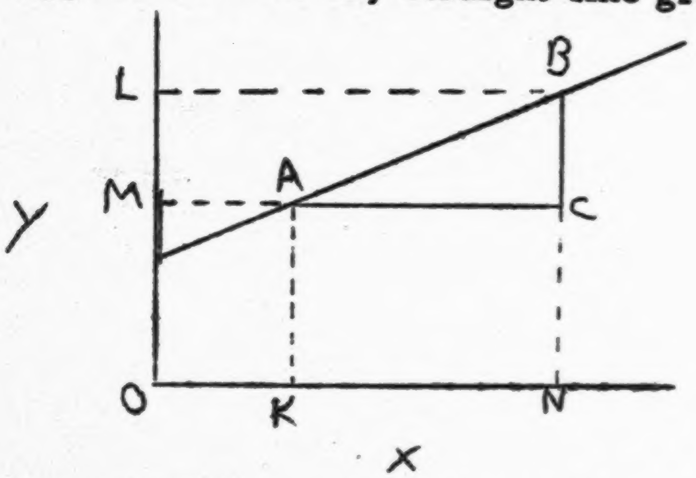
The whole idea of speaking of *the* slope of a straight line breaks down if different people get different slopes. Where do we go from here? Actually, you have run into this very problem several times before, though it appeared before in different disguises. For instance:

How can we speak of *the* spring constant, k, of a certain spring if k means "extension/weight", and its value depended on what weight you happened to use? Answer: the ratio, "extension/weight",is always the same for a given spring. (Experiment 11)

How can we speak of *the* density of aluminum if density means "weight/volume" and its value depended on what piece of aluminum you happened to use? Answer: the ratio, "weight/volume" is always the same for any piece of aluminum at all. (Experiment 15)

In Experiment 10, you found that the ratio of "height above the ground" divided by "distance along a ramp", for a given *ramp angle, is constant.* This means, in terms of the drawing above that $\overline{BC}/\overline{AB}$ and $\overline{EF}/\overline{DE}$ are equal, as is any other similar ratio for whatever points on the line you choose. In fact, we saw an argument (pages 71-72) showing that this ratio is the same, for a given ramp angle, regardless of what points you choose. Notice that this argument was directed to showing that "height/distance *along* the ramp" is a constant. It would be a good idea for you to go back and use the same argument (with only slight modifications) to show that "height/*horizontal* distance" is also constant. This ratio, too, is constant for any given angle that the line makes with the horizontal.

Now suppose we have any straight-line graph like this:



We choose any two points on the line, say A and B, and draw the usual lines AC and BC. You now know that, for any choice whatever of A and B, the ratio $\overline{BC}/\overline{AC}$ for this line is constant and is called the slope of the line. Suppose we extend BC to the x-axis, hitting the axis at N, and extend AC to the y-axis, hitting it at M. Also draw BL parallel to the x-axis and AK parallel to the y-axis. Could you plot the point A if its x and y are given? Certainly you can. Now backwards: can you find the x and y of a point if you are shown the point? Certainly you can! The x and y of the point A are respectively $\overline{OK}$ and $\overline{OM}$; and the x and y of the point B are respectively $\overline{ON}$ and $\overline{OL}$. Be sure you see that:

for point A: $x_A = \overline{OK}$ and $y_A = \overline{OM}$

for point B: $x_B = \overline{ON}$ and $y_B = \overline{OL}$

Next, we shall compute the quantities $\overline{BC}$ and $\overline{AC}$ (look at the drawing), whose values we need to find the slope of the line. You can see that $\overline{BC} = \overline{BN} - \overline{CN}$, can't you? But $\overline{BN}$ is equal to $\overline{OL}$ and $\overline{CN}$ is equal to $\overline{OM}$. Do you see why? But you just saw above that $\overline{OL} = y_B$ and $\overline{OM} = y_A$. We then have

$$\begin{aligned}\overline{BC} &= \overline{BN} - \overline{CN}\\ &= \overline{OL} - \overline{OM}\\ &= y_B - y_A\end{aligned}$$

You should also now be able to see that $\overline{AC} = \overline{MC} - \overline{MA} = \overline{ON} - \overline{OK} = x_B - x_A$. We now have shown that "$y_B - y_A$" is another name for $\overline{BC}$ and that "$x_B - x_A$" is another name for $\overline{AC}$. Then we have from the definition of slope that

$$\text{slope of line AB} = \frac{y_B - y_A}{x_B - x_A} \qquad (5)$$

Remember that A and B were selected in no particular way: they were any points on the line AB. Equation (5) then says that the slope of any straight line may be found by

(1) choosing any two points on the line;
(2) finding the difference between the y's of the points;
(3) finding the difference between the x's of the points;
and (4) dividing the y-difference by the x-difference.

All you have to be sure of, in applying this 1-2-3-4 recipe for finding the slope, is that the two points you use to find the x-difference are the same two points you used to find the y-difference, and that you subtract in the same direction both times.

Scientists usually use the symbol Δ to represent the difference between two quantities. Δ is a capital letter of the Greek alphabet, is pronounced "delta", and is the Greek equivalent of our letter D, meaning difference. For instance, Δx means a difference in two x's, Δy means a difference between two y's, ΔT might mean a difference between two temperatures. The symbol Δ is an exception to the rule that two symbols written together is an instruction to multiply. Δ is not a symbol for a quantity and therefore " Δ times something else" doesn't mean anything. Δ always means a difference. When you see two Δ's used in the same expression, you must always remember that the two differences they represent must be "corresponding differences." If I talk about Δy and Δx at the same time, this means that the "difference in y's" is worked out for the same two points as the "difference in x's." With this understanding of how we shall use the symbol, Δ, we can now write equation (5) more compactly as

$$\text{slope of a line} = \frac{\Delta y}{\Delta x} \qquad (6)$$

Remember: when you work out Δy and Δx to find the slope, you mustt find Δy for the same points that you use to find Δx.

You might notice that we can symbolize our definition of velocity (or speed) by using the compact Δ notation. On page 118, we defined velocity as (the distance the body travels) divided by (the time required to travel that distance). Let us write this definition as an equation.

$$\text{velocity} = \frac{\text{distance the body travels}}{\text{time required for the trip}} \quad .$$

In turn, we have defined distance as the difference between two positions. The "distance the body travels" is then the difference between its position at the end of the timing interval and its position at the beginning of the timing interval. [illegible] If we let p stand for position, then Δp is the change in position or distance traveled. The "time for the trip" is evidently the difference between the time at the end of the interval and the time at the beginning of the interval. We can represent this difference as Δt. Now you can see that Δp and Δt are "corresponding differences" in that they are measured over the same interval; hence we may use them in the same expression. The definition of velocity above then becomes very simply

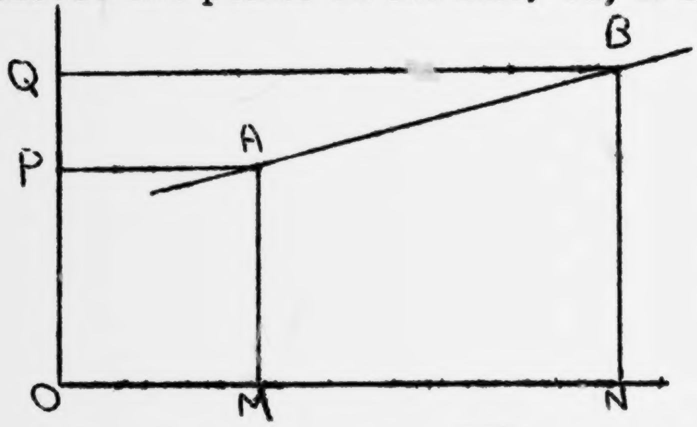
$$\text{velocity} = \frac{\Delta p}{\Delta t} \quad .$$

4. Slope of a Linear Function

By this time you know that any straight-line graph may be represented by a linear function, $y = a + bx$, where a and b are constants depending on what line you are talking about. Keep in mind that, once a and b are fixed, whatever they are, you are talking about one and only one straight line.

Now remember exactly what the relationship between the line and the equation is. Every point on the line has some pair of x and y. But you cannot choose any old x and any old y you please. Once you have chosen an x, then there is only one point on the line that has this x; and the point has only one y. So choosing an x automatically fixes a y: y is a function of x. But whatever the y and x might be, you can always calculate the y that goes with a chosen x by putting that value of x in the equation $y = a + bx$ and calculating the value of "$a + bx$." In other words, for every point on the line, the x and y of that point satisfy the equation $y = a + bx$. See?

Let us choose two points on the line, say A and B, whose x's are Δx apart. In the figure, then, $\Delta x = \overline{MN}$, and we can write



For the point A:: $y_A = \overline{OP}$ and $x_A = \overline{OM}$

For the point B: $y_B = \overline{OQ}$ and $x_B = \overline{ON}$

But the equation $y = a + bx$ is satisfied by every point on the line. Hence, for example,

$$y_A = a + bx_A.$$

We have other names for x_A and y_A as given above on the line "For the point A." If we place these values of x_A and y_A in the equation, we get

$$\overline{OP} = a + b \times \overline{OM} \qquad (7)$$

You should be able to see that substituting similarly in the equation, $y_B = a + bx_B$, gives

$$\overline{OQ} = a + b \times \overline{ON} \qquad (8)$$

Try it yourself!

Next, let us find an expression for $\overline{OQ} - \overline{OP}$. You can see from equations (7) and (8) that

$$\overline{OQ} - \overline{OP} = (a + b \times \overline{ON}) - (a + b \times \overline{OM}). \qquad (9)$$

Look at the last part of equation (9). We are subtracting the sum of a and b x $\overline{ON}$. You know that subtracting the sum of two numbers is the same as subtracting each number individually. Therefore we can rewrite equation (9) like this

$$\overline{OQ} - \overline{OP} = a + b \times \overline{ON} - a - b \times \overline{OM}.$$

Right away you see on the right hand side that a and -a appear, and may of course be canceled out. Do you remember why? Then you have

$$\overline{OQ} - \overline{OP} = b \times \overline{ON} - b \times \overline{OM}. \qquad (10)$$

Look at the right-hand side of this equation. It brings up a principle that we used before and will have occasion to use again. It is an arithmetical property of numbers that multiplying the sum of two number by a multiplier gives the same result as multiplying each of the numbers separately by the multiplier and then adding. In symbols,

$$a(b + c) = ab + bc.$$

This property of numbers is called the distributive principle, and applies to subtraction as well as to addition. We used the distributive principle on page 101, and wish now to use it again.

The right-hand side of equation (10) consists of the difference of two terms, each multiplied by the same number, b. Using the distributive principle, we can rewrite the right-hand side of equation (10) like this:

$$\overline{OQ} - \overline{OP} = b(\overline{ON} - \overline{OM}).$$

Now if you return to the drawing on page 127 you will notice that $\overline{OQ} - \overline{OP}$ is $\overline{QP}$ and $\overline{ON} - \overline{OM}$ is $\overline{MN}$. We therefore have

$$\overline{QP} = b \times \overline{MN}.$$

We can solve this equation for b (Do it yourself!) and obtain

$$b = \overline{QP}/\overline{MN}.$$

But now do you see that $\overline{QP}$ is the amount by which y changes in going from the point A to the point B? This means that $\overline{QP}$ is merely what we have previously called "Δy"; and similarly $\overline{MN}$ is just Δx. We therefore can rewrite the last equation above as

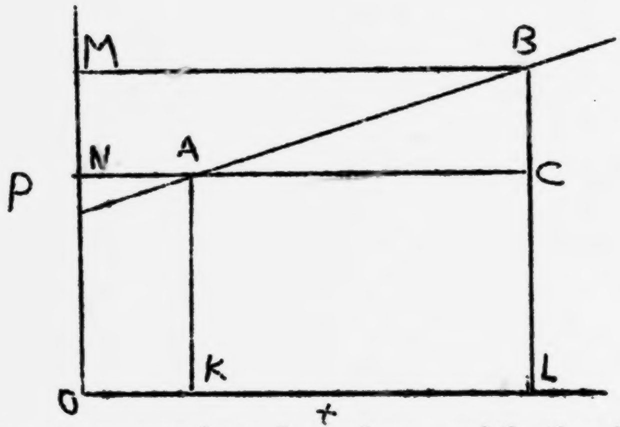
$$b = \Delta y / \Delta x. \qquad (11)$$

Compare this equation with equation (6). You will remember that we *defined* the slope of a line in accordance with our intuitive feeling of what the word "steepness" means, arriving at the fraction "$\Delta y/\Delta x$" as a reasonable definition. Earlier, we had learned that "y = a + bx" is an equation that represents any straight line at all. Now we find, by logic alone, on comparing equations (6) and (11), that the constant b in the equation for a straight line is nothing more nor less than the slope of that line!

5. Velocity and Slope

Let us now return to the graphs you made in Experiment 21. You have three straight lines, each of which therefore can be represented by an equation of the form "y = a + bx", where, of course, the a and b may be different for each line. You already know, in fact, that the a's are different; for a tells where the line crosses the vertical axis.

In the particular case of your graphs of position vs. time, we have noted before the convenience of using p to represent position (the quantity plotted vertically), and t to represent time (the quantity plotted horizontally). Instead then of writing "y = a + bx" as the equation of one of these lines, it will be more appropriate to write "p = A + Bt". Now choose any two points on the topmost line on your graph. Choose them so they are well-separated, and label the lower one A and the upper one B. (You are actually to do this; not just imagine it's being done!) Draw lines through A and B parallel to both axes as in the accompanying drawing, with the intersections as labeled there. Notice that point A represents the body when the time is $\overline{OK}$ and the position of the body is $\overline{ON}$, and the point B when the time is $\overline{OL}$ and the body's position is $\overline{OM}$. The time elapsed in going from A to B is therefore $\overline{OL}$ - $\overline{OK}$ and the distance traveled in that time is $\overline{OM}$ - $\overline{ON}$. But $\overline{OL}$ - $\overline{OK}$ is simply $\overline{KL}$ and $\overline{OM}$ - $\overline{ON}$ is simply $\overline{MN}$. Thus $\overline{MN}$ is the distance traveled by the body in the time interval $\overline{KL}$. By our definition of velocity, therefore,



$$v = \Delta p/\Delta t = \overline{MN}/\overline{KL}. \qquad (12)$$

From this equation, calculate the velocity of the falling ball for each of the three curves you plotted in Experiment 21. Follow the diagram above. After choosing two points A and B, draw the lines AC and BC and measure $\overline{BC}$ and $\overline{AC}$. Since $\overline{BC} = \Delta p$ and $\overline{AC} = \Delta t$, you can calculate the velocity. Record the calculated velocities at the bottome of Table I. Is your supposition borne out that a given size ball of a given material will fall through a given oil column always at the same velocity?

Now let us go back and apply equation (11) to the equation of motion at constant velocity, $p = A + Bt$. Equation (11) says that the slope of a line is the constant, b, in the equation, "$y = a + bx$", of that line, and is also given by the fraction, $\Delta y/\Delta x$. Applied to our particular case, equation (11) says that the slope of a line is the constant, B, in the equation, "$p = A + Bt$", of that line, and is also given by the fraction, $\Delta p/\Delta t$. In other words, $B = \Delta p/\Delta t$, when you are talking about the line, $p = A + Bt$. If you look at equation (12), you will see that the velocity of a body is also $\Delta p/\Delta t$. We have this interesting and important conclusion: For a body traveling at constant velocity, the position of the body is given by "$p = A + Bt$" where B is the velocity of the body.

We now can attach special significance to both the constants, A and B, in the equation, $p = A + Bt$. We saw earlier that A is the value of p when $t = 0$; that is, A is the position of the body at zero time. We might call this position, p_o. Now we learn that B is the velocity of the body. We then have the analytical expression in general,

$$p = p_o + vt, \tag{13}$$

where p is the position of the body at any time, t, and p_o is its position at time zero, and v is its velocity.

It is only fair to say that we could have arrived at equation (13) with far less labor. You could reason as follows: If the body has position, p_o, at time zero, then its position at a later time, t, will be p_o plus the distance it moves during the time, t. If its velocity is v, then the distance it moves in time, t, is vt; hence its position at time, t, will be $p_o + vt$. This three-line derivation is perfectly rigorous, but our purpose in using the longer derivation involving the general ideas of slope, linear functions, graphs of linear functions, Δ's, etc. was to develop your feeling for those ideas as well as to arrive at equation (13). We will use those general ideas again.

Now back to Experiment 21. You calculated the velocities of fall for the three runs and found these velocities to be identical (within experimental error). According to equation (13), the velocity is the slope. All three curves should have the same slope. Do they? The fact that the three curves are parallel is a reflection of their all representing motion at the same velocity.

By the same token, curves representing motion at different velocities will have different slopes. The motion would still be represented, of course, by equations like (13) but the values of v and hence the slopes will be different. Now make calculations of the velocities of the three (or four) runs of Experiment 22. Do it in the same way you calculated the velocities in Experiment 21, and record them at the bottom of Table II. Compare these velocities with your judgement of the slopes of the curves. Does it seem reasonable to you that greater velocity should mean greater slope?

What is the slope of a horizontal line on the graph? How much does p increase for any interval of t on a horizontal graph? What then is the value of Δ p for any interval? What then is the value of the slope? What kind of "motion" does a horizontal line represent? Do slope and velocity agree?

Equation (13) gives p as the sum of two terms, p_o and vt. You learned long ago that you can add two quantitites only if they have the same units. Can you show that p_o and vt have the same units?

You remember that (page 124) we could speak of the slope of a straight line because a straight line has always the same steepness; that is, its slope is constant. You learned that the falling balls in the last two experiments gave you straight lines when you plotted p versus t. These curves then have a constant slope, since all straight lines have a constant slope. You also learned that the slope of a p vs t curve is the velocity. It follows therefore that the fact that you got straight lines when you plotted p vs t proves that the motion of the balls was under constant velocity.

6. Accelerated Motion

Up to this point we have confined our attention to uniform motion; that is, motion at constant speed (or velocity) along a straight line. Any motion that takes place other than along a straight line at constant speed is called "accelerated motion." We will continue to restrict ourselves to motion along a straight line, but will now consider motion in which the speed is not constant. You may never have noticed that a rock dropped from the roof of a house does not fall with constant speed. It falls at first very slowly, then picks up speed and moves faster and ever faster until it hits the ground. Because of friction against the air, a rock dropped from a very tall building or an airplane would eventually reach a constant velocity. Using a small ball, and an oil where the friction is much greater than with air, the constant speed is reached after dropping only a centimeter or two. This was the idea behind the last two experiments.

Dropping a rock through the air then is a good example of accelerated motion. The motion here is much too fast for us, however, to make measurements on the motion as we did for the ball falling through oil. But it is also true that a ball rolling down a ramp in air behaves much like a free-falling rock, except that everything is slowed down to a point where you can make convenient measurements.

In Experiment 23, you will study the motion of a ball rolling down a ramp as an example of accelerated motion. Do you know what the accelerator on an automobile does? The accelerator is the gas pedal, and by pushing down or letting up on the accelerator, you can make the car go faster or slower. In other words, the accelerator allows you to change the speed of the car. "Accelerate" is a verb meaning to "change the speed of" something. We use the word the same way here. Sometimes "accelerate" is used to mean only "increase the speed of", but we shall use it to refer to any kind of change in speed, not only an increase.

Accelerated motion then is motion whose velocity changes. Suppose you observed a moving body and wished to determine whether its motion was accelerated. How might you do it? Well, since accelerated motion is merely motion whose velocity changes, here is one sensible thing to do: You could choose any two points in the path of the body and measure its velocity at both points. If the velocity is not the same at both points, you may certainly conclude that the motion is accelerated.

Notice that a body moving with constant velocity has no acceleration. It might move forever with never any change in its velocity; no matter how fast or how slowly it's moving, then, it has no acceleration. This observation is the germ of a physical idea. For if we say "the body has no acceleration," we are tempted to reword the statement as "the body has an acceleration of zero." But zero is a number, and right away the physical scientist would say, "I can imagine a body that has zero acceleration, and zero is a number. I can also observe a body that does not have zero accleration. I wonder whether acceleration is one of those qualities that can be expressed numerically. I haven't yet defined exactly what the word 'acceleration' shall mean, though I have an intuitive feeling that it ought somehow to have something to do with change in velocity. Can I define it in such a way that acceleration becomes a measurable quantity ?"

We can kick this idea around a little further. Imagine two cars standing side by side, motors running, ready to begin a drag race. At the same instant the drivers step on the gas. One car takes 10 seconds to reach a speed of 60 miles an hour starting from rest, and the other car requires 15 seconds to reach that speed. Both cars accelerated, for both changed their speeds from zero to 60 mi/hr. The first car changed its speed from zero to 60 mi/hr more quickly than the second. We feel that the verb, "accelerate", ought to contain in its meaning something that would allow us to say that the first car accelerated more than the other. If we bother to define the word acceleration precisely, then, we would like it to be defined in such a way that the first car will have a greater acceleration than the second.

But both cars had the same "change in velocity", for both started with zero velocity and speeded up to 60 mi/hr. Once again you see that something more is involved here than simply "change in velocity." What we are really concerned about is not how much the velocity changes, but how rapidly it changes.

You should now do Experiment 23. After you are finished we will have a lot of

Points to Discuss in Class

Do the plotted points fall on anything that looks reasonably close to a straight line? Since the point, (position = 0, time = 0), lies on the graph, your curve passes through the origin. Would it be fair to conclude that position (or distance) is proportional to time?

If distance traveled were proportional to time, you now know that the equation of the curve would be "p = vt", where v is the constant velocity of the body. Since the curve is not a straight line, then, your experiment shows that the body does not travel with constant velocity. The motion is therefore, by definition, accelerated.

Draw the curve, position vs time, smoothly as best you can through the points. It is probably that the points will not fall all on a smooth curve. Try to draw a curve through the points in one single sweep, placing it as usual in such a way that roughly as many off-points lie on one side of the curve as on the other.

Now fill in the third column of the data sheet for Experiment 23 with "smoothed values of the time." You recognize that the points on your graph are "off" because of experimental errors incurred by the difficulty in making precise measurements on such a rapidly moving body. If you make enough observations, however, you have a feeling (here is that intuition again!) that you probably made as many mistakes giving readings too high as too low. This is another aspect of the point discussed on page 21 about feeling that an average is probably better than one reading alone. You therefore draw your curve so that some points lie above and some below it, believing that the "true" curve lies comfortably "down the middle". We are now saying that we believe the curve itself gives "better" values for the "times of passage" than the ones actually observed. We are saying, that is, that when the "down the middle" curve passes below an observed point, that the observed point is probably "too high" because of experimental error; and that an error-free measurement would have placed the point close to the curve. We are saying that the curve, which is based on many readings of the same function, is more reliable than any one point.

If, then, you wanted to know the time of passage to, say, 160 cm, it would be better to read the smooth graph than to take the actual measured point. Do this, reading the "smoothed" values for times of passage for every position listed in column one of the data sheet. These smoothed values are generally regarded by scientists as more reliable than the ones actually observed by measurement. Notice that it is a method of finding an "average time of passage to 160-cm" that takes into account not only your measured values at 160-cm, but also your measurements at other positions as well.

Does the curve have a constant slope? Can you tell whether the curve -- as you move to the right, or as time goes on -- becomes increasingly or decreasingly steep? You have learned that, when you plot position vs. time, the slope of the curve is the velocity. From the slope of the curve only, does the velocity appear to be constant? From the slope of the curve only, does the velocity appear to be increasing all the time the ball is rolling?

Now (this is review) you know that you can find the position of the body at any time by reading the graph for that time. Moreover, any given t has associated with it one and only one point on the graph, hence one and only one p. In other words, given t, you can always find one and only one p. From the definition of function, we say that p is a function of t. But notice that the slope also changes with t. Just as each t leads to exactly one p on the curve, so also does each t lead to exactly one slope of the curve. In other words, the slope is also a function of t; because if you are given any t, the slope of the curve at that t is fixed by the curve whether you like it or not.

But remember that (page 130) the slope of the curve p vs t is the velocity of the moving body. Your graph, we now see, shows that not only the position of the body but also the slope -- and therefore the velocity of the body -- is a function of the time. You should understand that this conclusion is quite in accord with your observation: you noticed that the ball speeded up more and more as it rolled downhill; that at each different instant during the roll, its velocity was different, depending upon what instant you are talking about. So your observation alone tells you that the ball's velocity is a function of the time. The preceding discussion was only to call your attention to the fact that this information is revealed by the graph, too, if you know how to read the graph.

We therefore have good reason to believe that the velocity of the rolling ball is a function of the time. Can we find out what function it is? What do we already know about the velocity? We know that at time zero the velocity was zero, for we started counting time when the ball was at rest and we made sure that the ball was allowed to pick up speed by itself. We also know that the velocity is an increasing function of the time, for the slope of the curve becomes steadily greater (that is, the curve becomes steadily steeper) as we go to greater times. Thus we know beforehand that thecurve will pass through the origin and will slope always upward to the right. Does this suggest anything to you?

Now let us try to find the velocity for some certain time, If you look at your table of data, you will remember that you have in column one a list of positions of the ball and in column three a list of the times at which the ball was in those positions. For instance, the first line of the table tells you that the ball was at a position of 0 cm at a time of 0 seconds. The second line tells you that when the ball was at a position of 40 cm, the time was 2.4 seconds (Near there, anyway; the exact time you got depends upon how high the end of the ramp was propped up, how hard is the wood of which the ramp was made, and several other things.). This means that the ball traveled a distance of 40 cm in 2.4 seconds. We know that its velocity was not constant over this interval. But suppose we did have a body moving at constant velocity that covered this same distance of 40 cm in the same time of 2.4 seconds. What would that constant velocity have to be? You should be able to calculate that a body traveling 40 cm in 2.4 seconds at constant velocity would have to travel with a velocity of about 16.7 cm/sec. We call this constant velocity by the name of "average velocity." "Average velocity" merely means the constant velocity that would "do the same job in the same time" as some other body not moving at constant velocity.

In terms of our Δ notation, the "difference in position in moving from a position of zero to a position of 40 cm is $\Delta p = 40$ cm. Correspondingly, $\Delta t = 2.4$ sec. When we calculated the average velocity above, then, we actually used the definition of velocity given on page 129 :

$$v = \frac{\Delta p}{\Delta t} = \frac{40\ \text{cm}}{2.4\ \text{sec}} = 16.7\ \text{cm/sec}$$

So we are not really introducing anything new here; we are merely extending our earlier definition of velocity as $\Delta p/\Delta t$ to the case when the velocity is not constant.

Now the velocity at the beginning of the interval was zero, because the body was at rest. The average velocity over the interval was 16.7 cm/sec. Do you see therefore that the velocity at the end of the interval (that is, at the instant that the ball was passing the 40-cm mark) must have been greater than 16.7 cm/sec? This must be the case, because if the velocity started out less than 16.7 cm/sec in order to get the average up to 16.7. Thus the body started out slower than 16.7 cm/sec, it must have ended up greater than 16.7 cm/sec in order to get the average up to 16.7. Thus the body started out slower than 16.7 cm/sec and ended up faster than 16.7. Do you see that the body must sometime in between have had a velocity of exactly 16.7 cm/sec? This is the principle of continuity, which says that if the body's velocity changed from something less than 16.7 to something greater than 16.7, it cannot have "skipped over" any velocity in between. Of course we don't know exactly where and when it had this velocity, but we do know that at sometime between 0 seconds and 2.4 seconds, and somewhere between 0 cm and 40 cm, it did have this velocity. With very little justification other than that the time and place must be somewhere between, let us take the time and place of the average velocity as midway in the interval. That is, let us say that the body had a velocity of 16.7 cm/sec when it was halfway betweeen 0 and 40 cm and halfway between 0 and 2.4 seconds. To repeat: though we don't know that it is exactly correct, we do know that the body had a velocity of 16.7 cm/sec somewhere close to a position of 20 cm and a time of 1.2 sec.

To sum up: During the interval over which the ball rolls from the 0 to the 40-cm mark, its position changes by $\Delta p = 40$ cm; time increases by $\Delta t = 2.4$ sec. The average velocity during this interval is $\Delta p/\Delta t = 16.7$ cm/sec. The approximate place and time at which the body had exactly this velocity are 20 cm and 1.2 sec. Refer now to the right-hand portion of Table I in Experiment 23. The data in the preceding two sentences are to be entered on the first line of this table. $\Delta p = 40$ cm is already entered; Δt will be somewhere near 2.4 sec, depending on what you measured for the time of passage to the 40-cm mark. "v" means "average velocity." (Physical scientists quite commonly denote the average value of a variable by placing a bar over the symbol for the variable. You read v as "vee bar".) Calculate your average velocity for the first interval and enter it in this column. The next two columns contain the midway points, both in distance and time. In distance, of course, the midway point is 20 cm; in time, the midway point you will have to calculate yourself. It will be close to the 1.2 sec used in the example above.

You should now be able to fill in the rest of the right-hand portion of Table I. First, notice that the Δp's are all the same; namely, 40 cm because the positions listed in column one are 40 cm apart. Next, remember that the first Δt is the time interval fromthe zero mark to the 40-cm mark. You find it by subtracting the "time at 0 cm" from the "time at 40 cm." The next interval - 40 cm to 80 cm - begins at the time-of-passage for 40 cm and ends at the time-of-passage for 80 cm. The Δt is the difference between these times, which should be entered on the second line under Δt. $\bar{v}$, of course, is simply $\Delta p/\Delta t$. The midway position for the second interval is halfway between 40 cm and 80 cm; calculate this position and enter it on the second line under "Midway position." The midway time for the interval is halfway between the times at the beginning and end of the interval, both of which you get from column three. Calculate the midtime and enter it in the second last column. Now complete the table yourself.

The third last column of Table I now gives you the velocity the ball had at the time given by the second last column. These two columns therefore are a tabular representation of the functional relationship we were seeking -- that giving the velocity as a function of time. In the space on the lower half of the second work sheet, make a graph of velocity vs. time, velocity vertically and time horizontally. Do you notice anything especially to be remarked about this graph?

Your graph shows that in the case of a ball rolling downhill, velocity is proportional to time, for the graph is a straight line passing through the origin. Use a ruler to draw in the curve, again trying to place the straight line so that it steers up the middle of the plotted points.

You know now that the equation of this straight line must be "$v = at$", where v represents the variable velocity, t the time, and a is a constant. You also know that, if "$v = at$", then "$a = v/t$". Can you still show this? The fact that your graph was a straight line then shows that the ratio v/t is a constant. For each line of Table I, calculate the ratio v/t, using the interval-average velocity, $\bar{v}$, for the numerator and the midway time, t, for the denominator. Record these ratios in the last column of Table I. Is this ratio reasonably close to constant? What are the units of this ratio?

The ratios you calculated for v/t are nearly constant (within experimental error), and will probably come out to be somewhere around 15 to 20 cm/sec^2. (You read this as "20 centimeters per second per second" or "20 centimeters per second squared".) If your ratio came to, say, 17.0 cm/sec^2, then the functional relationship between velocity and time that you were looking for is

$$v = 17.0\ t.$$

If the units of "17.0" are cm/sec^2 and those of t are seconds, what will be the units of v? Is this reasonable?

We have noted many times before that in a proportionality equation like $y = kx$, the proportionality constant, k, often has some special physical significance. What is the meaning of the constant, a, in the equation, $v = at$? In order to get a feel for its meaning, notice first that the velocity keeps changing (in our case, increasing) with time. The graph slopes always upward to the right. Can you judge the relative appearance of two graphs of the form $v = at$, one of which represents a body whose velocity changes only very slowly and the other of which represents a body whose velocity changes very rapidly? Which of these two straight-line graphs will have the greater slope? Do you see that the line with the greater slope goes with the body whose velocity changes more rapidly?

Now, you have seen (at the very end of Section 4 in this unit) that the constant k in the equation $y = kx$ is simply the slope of the graph of that line. Similarly, the constant, a, in your equation, $v = at$, is simply the slope of your graph. Furthermore, you know that the greater the value of a, the more steeply the curve climbs upward to the right. But from the latter part of the preceding paragraph, you saw that the more steeply the curve climbs, the more rapidly its velocity changes. Recall now that (page 132 , just before doing Experiment 23) that we were looking for a way to define the term "acceleration" numerically. We agreed that "acceleration" should refer to how fast the velocity changes. Maybe we now have an acceptable definition of the word. If the velocity of a body is proportional to the time, then velocity and time are related by an equation, $v = at$, where a is a constant, namely the slope of the curve whose equation is $v = at$. If this curve is horizontal, then $v = 0$ everywhere, doesn't it? Also, then the slope, or a, is zero; this means that the velocity never changes. That is, if $a = 0$, the velocity stays constant or there is no acceleration. If the curve is steep, the velocity changes slowly and a is small. It looks as though we could take this quantity a to be the quantity that we wanted to call "acceleration." In fact, when the velocity is proportional to the time, we will define the quantity a to be the acceleration.

Since earlier (page 126, equation (6)) we defined the slope of a straight line as $\Delta y/\Delta x$, you can see that our definition amounts to this:

$$\text{acceleration} = \Delta v/\Delta t$$

when that ratio is constant. You will recognize that "$\Delta v/\Delta t$" is simply a symbolic way of saying "change in velocity divided by the time interval over which the change takesplace." Keep in mind then that there must be a change in velocity for the acceleration to be other than zero. An interplanetary rocket traveling at 50,000 miles per hour has zero acceleration if its velocity remains at 50,000 mph.

In the case of the ball rolling downhill, you have shown that the acceleration is constant, because the velocity is proportional to the time. You have shown, in other words, that, for a moving body,

The velocity is proportional to the time.

and

The acceleration is constant.

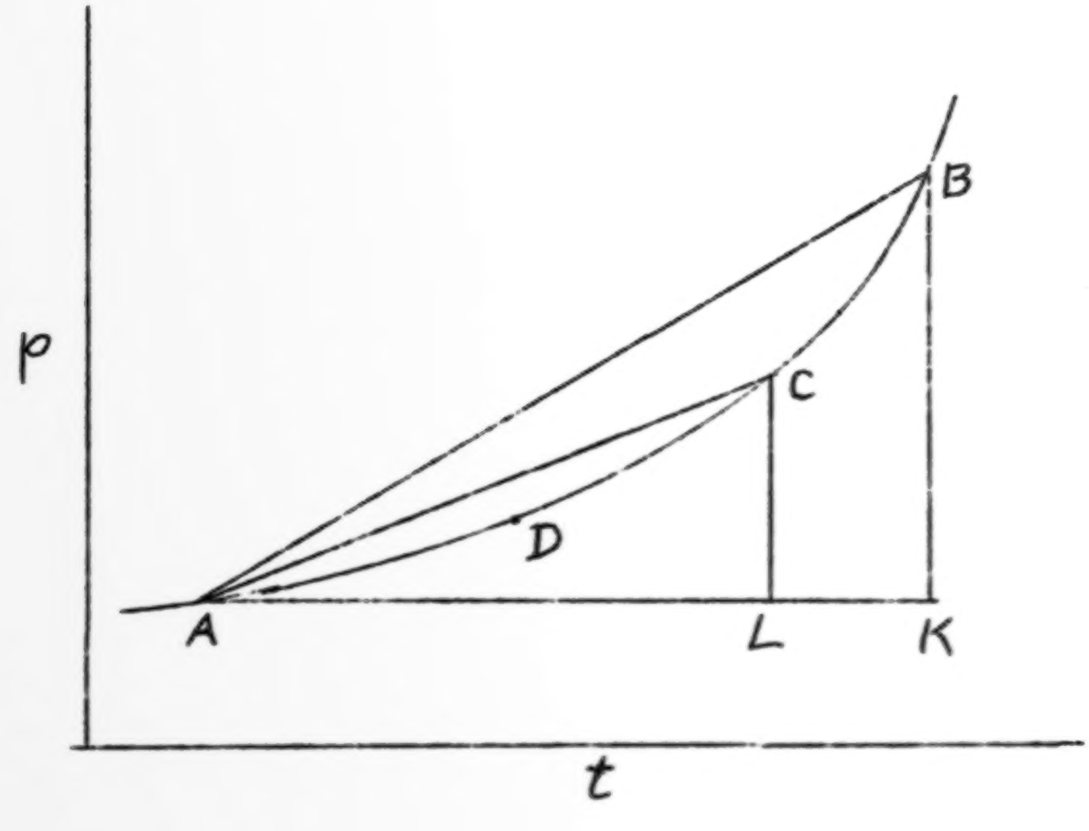
are two exactly equivalent statements.

7. Tangent to a Curve

You will remember that we defined velocity (page 129) as $\Delta p/\Delta t$ under conditions of constant velocity. We defined "average velocity for an interval" also as $\Delta p/\Delta t$ for that interval. Notice then that we have no "unconditional" definition for velocity. We say that $\Delta p/\Delta t$ is the velocity if it is constant, or it is the average velocity if not. Is it possible to define what is meant by velocity (not average velocity) even when the velocity is itself changing?

Notice that it is possible to define what is meant by position even when a body's position is changing -- that is, even when the body is moving. We can easily conceive of the idea of "instantaneous position" -- that is, the position of the body at one certain instant. We feel that it ought also to be possible to tell how fast a body is moving at some certain instant -- instantaneous velocity -- even when the body is moving. As a matter of fact, this is exactly what the speedometer of an automobile does. What is meant by instantaneous velocity? Look back at the graph you made in the upper part of the second work sheet for Experiment 23. You plotted position vs time in this graph, and obtained a curve that is not a straight line. If the graph of position vs time is a straight line, then you learned, following Experiments 21 and 22, that the constant slope of that line is the velocity. Does it seem reasonable to you that if the slope is not constant, then we could still define the changing velocity as being the changing slope? This certainly is reasonable, because the more steeply the curve rises, the more rapidly its velocity changes.

Let's look at a picture that may help you see the point. The curved line in the diagram below may be thought of as a portion of your graph of p vs t in Experiment 23. Suppose that we want the slope of this curve at the point A.



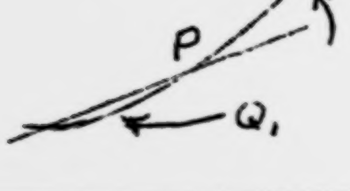
Referring to the definition of slope on page 126, we see that we need to find $\Delta p/\Delta t$. We choose some other point on the curve, say B, and lay off the two Δ's, $\Delta p = \overline{BK}$ and $\Delta t = \overline{AK}$, as we did so many times before. The fraction $\Delta p/\Delta t$ is then the slope; --- but the slope of what? Actually $\Delta p/\Delta t$ is the slope of the line AB, which is clearly steeper than the slope of the curve at A, and not nearly so steep as the curve at B. As far as the curve is concerned, this fraction $\Delta p/\Delta t$ does not give us the slope at either A or B, but rather at some point between A and B. Try estimating the point where the curve has the same slope as the line AB.

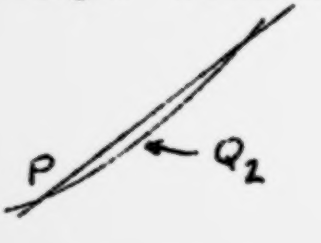
Suppose we had chosen some point C instead of B as the second point, where C is closer to A than B is. Then we could calculate a new $\Delta p/\Delta t$ which would equal $\overline{CL}/\overline{AL}$. But this would give us the slope of the line AC, which again is the slope of the curve at neither A nor C, but at some place between them. Try estimating the point where the curve has the same slope as AC. We could choose as the second point a still closer point, say D, to A than either B or C; and the calculated $\Delta p/\Delta t$ would again be, not the slope at A, but at some point between A and D. About where, would you say?

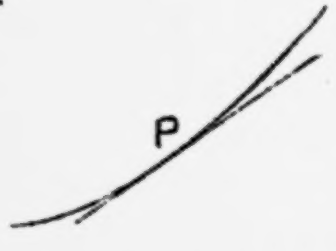
Do you see that each time you choose a new second point, closer to A than the last one, for measuring the two Δ's, you get a new slope? This new slope is not the slope at A, but the slope at some point on the curve closer to A than the last one. If you keep choosing points closer and closer to A, then the point of the curve where the slope is the same as the straight line gets closer and closer to A.

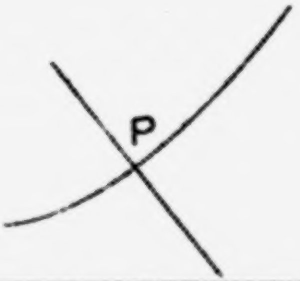
The contents of the last two paragraphs are intuitive. Have you noticed that we have been talking about the "slope of a curve at some point" without ever having said exactly what we mean by that expression? We do, however, have an intuitive feeling about what we would like to have the expression mean. Suppose someone gave you a yardstick, led you to a curved sliding-board, and pointed to one spot on the side of the sliding board. Could you tilt the stick so that the stick had the same slope as the sliding-board at that point? Would everyone agree on exactly how much the stick should tilt in order to have the same slope as the curve? Or, to put it another way: if someone disagreed with your idea of the right tilt, how would he go about proving you wrong? You can't prove someone wrong until you agree on a definition of what is right.

The curious thing is that most people would agree on what is meant by the slope of a curve, at least to the extent of judging when a curve at some given point is equally steep with an adjustable straight stick. What we must try to discover is the unconscious basis that people use for their judgement without having a definition. We can get at the matter like this: Suppose we have a given curve like the one in the sketches below, and a given point on the curve, like P. In one of the four sketches, the straight line and the curved one have the same slope. Which one?









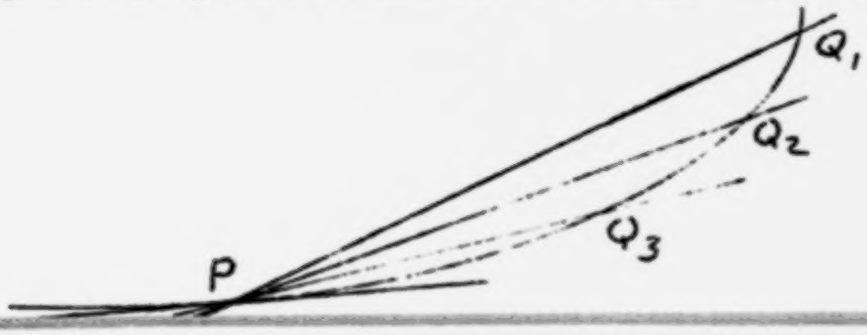
In the first sketch, the straight line cuts the curved one in two points. As we discussed before, you feel that at some point between the two intersections, the curve and the line have the same slope, but not at the point P itself. The curve looks as though it has the same slope as the line at about the point Q_1. Now let us keep the line pinned at P, but free to rotate like the propellor of an airplane. Rotate the line in the direction of the little arrow so that the line, still pinned to the curve at P takes the position of the second sketch. Again the straight line cuts the curve in two points. Again we feel that the line does not have the same slope as the curve at P but more like that at Q_2. By rotating the line, we have moved the point where line and curve have the same slope from Q_1 to Q_2. Notice that Q_1 lies on the curve below P and Q_2 above. In other words, rotating the line around P from its position in the first sketch to that in the second caused the movable point Q to go from somewhere below P to somewhere above.

The principle of continuity suggests to us that at some time during the rotation, the movable Q must have passed through P. That is, somewhere between the positions of the line in these two sketches, the line had the same slope as the curve at P. We rotated it too far. How far should we have rotated it?

You probably see the idea by this time. As long as the line cuts the curve in two points, P and another one, the line will not have the same slope as the curve at P, but rather will have the same slope as some point between the two intersections. The only way we can arrange the line so its slope will be the same as at P is if we have the line touch the curve in only one point, as in the third sketch. A line that touches a curve in only one point is called a tangent to the curve at that point.

But wait a minute, you say. The fourth sketch shows a line that also touches the curve in only one point. Is the line in the fourth sketch also a tangent? If so, you can see that you can draw lots of similar lines that pass through P and cut the curve in only one point. The answer is no. A tangent not only cuts the curve in only one point, but also lies on one side of the curve only. Notice that in the first, second, and fourth sketches the line crosses over the curve at P, from one side to the other. In the third sketch, the line touches the curve at P without crossing. This is a tangent: a straight line that touches a curve at one point without crossing it. A line that crosses a curve is called a secant. (The word "tangent" comes from the Latin word tangens, which means "touching." The word "secant" comes from the Latin secans, which means "cutting.")

Notice that one way to think of a tangent is the following, patterned after the sketch below: First, draw some secant to the curve through P, say PQ_1. Now keep the line pinned to the curve at P and allow Q_1 to move along the curve



toward P, through the points Q_2, Q_3, etc. As you do this, the secant rotates around P. Finally, when Q comes right on top of P, the secant has become a tangent. Thus you can think of a tangent as the limiting position of a secant as one of its two intersections approaches the other.

Now finally we can define what we mean by the slope of a curve at a point: it is simply the slope of the tangent at that point. Again, you should notice that the last several pages do not prove that the slope of a curve at a given point is the same as the slope of the tangent at that point. This is a definition of what is meant by "slope of a curve at a point" -- a notion that we had not previously defined yet felt that we intuitively grasped. The long discussion preceding is to show you that this definition agrees with your intuitive feeling of what "slope of a curve" ought to mean.

Of course, if you are given a curve already drawn, you could use a ruler to draw a tangent to the curve at an assigned point just by using the judgement of your eye. This might be done in much the same way as we tried to adjust the tilt of the yardstick to that of the sliding-board a few pages back. This kind of judgement by eye, in fact, is often quite good. Let's try it on the data you obtained for Experiment 23. Do Experiment 24 now.

Points to Discuss in Class

What is the physical meaning of the $\Delta p/\Delta t$ that you calculated from your measurements? Remember that the Δp and the Δt that you measured were obtained from the straight-line tangents that you drew. Therefore $\Delta p/\Delta t$ is the slope of the tangent. But we agreed that "slope of tangent at a point" means "slope of the curve at that point." So the $\Delta p/\Delta t$'s that you found in Experiment 24 are actually the slopes of the curve at the points where you drew the tangents.

But recall now that the slope of a p vs t curve (page 130) is the velocity of the body at the point where the slope is measured. Therefore the $\Delta p/\Delta t$ that you measured and computed from the tangent you drew at p = 140 cm is actually the velocity, v, of the ball at the instant it passed the 140-cm mark. Compare the velocity obtained from the slope of the tangent with $\bar{v}$, the average velocity over a small interval surrounding the 140-cm mark. The two values -- approximate $\bar{v}$ and "exact" v -- should be nearly the same but not identical. Which one, $\bar{v}$ or v, gives the instantaneous velocity at the point?

When you measured $\Delta p/\Delta t$, you obtained the slope of the tangent, which is a straight line. Does it matter what interval you use for the Δ's when you measure the slope of a straight line? Then why were you told to choose the points A and B "at least 15 cm apart"? What avoidable error might arise if A and B were only, say, 1 cm apart?

In your opinion, is there any judgement involved in estimating the correct position of the ruler to make it tangent to the curve at the point P? Most people will agree quite closely on where the ruler should be positioned, but even one person will not always choose exactly the same position. The question comes up: Is there a way to find the slope of a tangent to a curve that does not require a judgement that may not always be reliable? There is, if you can find an analytical representation for the curve.

Do you have ananalytical representation for this curve? No. You know that p is not a linear function of t, but that's all you know at the moment. Thus you know that the ratio p/t is not constant. What would you suggest trying instead of p/t in the hope of finding some constant ratio. involving p and t? Think of it this way: If p/t were constant, the graph of p vs t would be a straight line and would have a constant slope upward to the right. Look at your graph of p vs t. It slopes upward with ever-increasing slope. This means that p increases "faster than t". Maybe p is proportional not to t but to something that increases faster than t -- maybe t^2. Try it. Compute t^2 for each line of the table, recording the values in the appropriate column. Then compute p/t^2. This ratio should be nearly constant. Calculate the average value of p/t^2, and call this constant, k.

That p/t^2 is a constant means, as you know, that

$$p = kt^2.$$

We have already seen that this ball rolling downhill moves with constant acceleration. Is there any relationship between the constant acceleration and the above constant, k? There is, but don't jump too quickly to a conclusion!

8. Derivative of a Function

You can always find the slope of a straight line: it is simply $\Delta y/\Delta x$. You can therefore find the slope of any secant to a curve that might be given to you. If you allow the secant to swing around one of its intersections as in the figure on page 140, you get a whole series of values of $\Delta y/\Delta x$, each of which is the slope of a secant that lies closer and closer to the tangent. If we could find the number that $\Delta y/\Delta x$ gets closer and closer to, then we would know the slope of the tangent. We will see that this can actually be done.

Imagine that the Δy's and Δx's were actually drawn in the sketch above. Perhaps it would be well for you to draw a curve on a piece of scratch paper and choose a point P on the curve. Then choose a series of Q's, each progressively closer to P than the last. Finally draw in the Δy's and Δx's for each Q. For Q_1, say, you can then measure Δy and Δx, and compute their ratio. Call this ratio $(\Delta y/\Delta x)_1$ for the point Q_1. You could do the same thing for Q_2, obtaining the ratio $(\Delta y/\Delta x)_2$. In this way you could get a series of $(\Delta y/\Delta x)$'s, one for each Q.

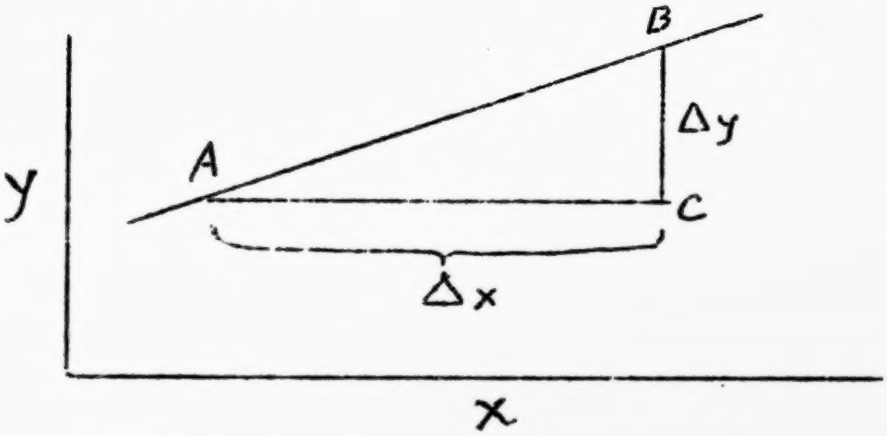
But notice that as the Q you choose gets closer and closer to P, the measured values of Δy and Δx get smaller and smaller, and more and more difficult to measure. In the limit when Q has come to coincide with P in fact, both Δy and Δx will be zero, and then we couldn't calculate the ratio $\Delta y/\Delta x$ anyway. The ratio would then be 0/0, which not only cannot be calculated but also is undefined. But remember that 0/0 can be defined if we want to. In this case we would want to define 0/0 in such a way as to fit snugly into the series $(\Delta y/\Delta x)_1$, $(\Delta y/\Delta x)_2$, $(\Delta y/\Delta x)_3$, etc., for only in that way could we satisfy the principle of continuity.

Now before going on, it is important that you understand the following: We want to find what value $\Delta y/\Delta x$ gets closer and closer to, as Q is made to come closer and closer to P, until in the limit Q coincides with P. But as Q comes closer and closer to P, Δx becomes zero. So another way to say the same thing is to say that we want to find the value of $\Delta y/\Delta x$ when Δx becomes zero. It may surprise you that we can actually find this limiting value. We can do it when we have an analytical expression for y as a function of x.

Suppose that, for example, y is a linear function of x:

$$y = a + bx. \tag{14}$$

Let us find the value of $\Delta y/\Delta x$ as Δx is allowed to become so small as to be zero. Remember that equation (14) applies to every point on the curve (in this case, of course, a straight line). Suppose we choose two points, A and B,



and draw the perpendicular lines, AC and BC, in the usual way that you are now so familiar with. The distance AC is what we have been call Δx and BC is what we have been calling Δy. Suppose we say that the point A is the point whose x-and-y combination is x_0, y_0. Then it must be true, since A is on the curve whose equation is (14), that

$$y_0 = a + bx_0. \tag{15}$$

Do you see that the x-value for the point B is $(x_0 + \Delta x)$, and the y-value for the same point is $(y_0 + \Delta y)$? But since the point B is also on the curve whose equation is (14), it must also be true that the x-and-y combination for B satisfies equation (14). That is,

$$y_0 + \Delta y = a + b(x_0 + \Delta x). \tag{16}$$

The last two equations give us alternate names for two quantities: one of them is y_0, and the other is $(y_0 + \Delta y)$. Therefore if we subtract the left-hand side of (15) from the left-hand side of (16), we will get the same result as when we subtract the right-hand side of (15) from the right-hand side of (16). Subtracting the left side of (15) from the left side of (16) gives

$$y_0 + \Delta y - y_0;$$

and doing the same for the right-hand sides gives

$$a + b(x_o + \Delta x) - (a + bx_o).$$

These two subtractions, we have just agreed, give the same result. Hence

$$y_o + \Delta y - y_o = a + b(x_o + \Delta x) - (a + bx_o).$$

You can see right away that "y_o" and "$-y_o$" can be crossed out on the left, and then we have

$$\Delta y = a + b(x_o + \Delta x) - (a + bx_o).$$

Now you will remember that subtracting $(a + bx_o)$ is the same as subtracting a and also bx_o individually. That is,

$$\Delta y = a + b(x_o + \Delta x) - a - bx_o.$$

And again you can see that "a" and "-a" on the right may be dropped out:

$$\Delta y = b(x_o + \Delta x) - bx_o.$$

Next, recall the distributive principle: that $b(x_o + \Delta x)$ is the same thing as $bx_o + b\Delta x$. Therefore

$$\Delta y = bx_o + b\Delta x - bx_o.$$

Again you can drop the "bx_o" and "$-bx_o$", and find that

$$\Delta y = b\Delta x.$$

From this last expression you can easily find that

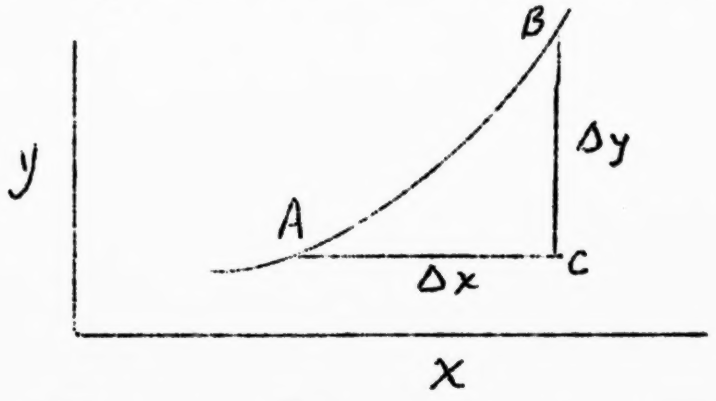
$$\frac{\Delta y}{\Delta x} = b. \qquad (17)$$

This is the same result we previously found (page 123); namely that for any linear function "$y = a + bx$", the slope, $\Delta y/\Delta x$, is simply b. In fact the reasoning we just now used is identical with the reasoning we used before. We have simply changed to a different set of symbols. The reason for doing it all over again was just to put you on familiar ground before we used the same procedure for a function that is a little more complicated than a linear function.

Notice that the choice of how big Δx was, in the argument above, was quite undecided. We never committed ourselves to any particular value for Δx, and hence equation (17) is true for any Δx whatever. We are especially interested in the case when $\Delta x = 0$, for which $\Delta y/\Delta x$ also, of course, is b. This independence of the value of $\Delta y/\Delta x$ upon Δx is not always the case.

It is a result, in fact, of y's being a linear function of x, and in general the value of $\Delta y/\Delta x$ does depend on how big Δx is. You have already seen that $\Delta y/\Delta x$ depends on how far apart the chosen points are when the curve is not a straight line, and does not depend on how far apart they are when the curve is a straight line. This conclusion, then, is not new, but it is worth recalling in this new context.

Suppose that y is not a linear function of k. Let us consider the case when $y = kx^2$. Again we consider two points, A and B, and the corresponding



Δ's, $\Delta x = \overline{AC}$ and $\Delta y = \overline{BC}$. The x-and-y combination for the point A we will again call x_o, y_o; and that for the point B we will again call $x_o + \Delta x$, $y_o + \Delta y$. Since both A and B lie on the curve whose equation is $y = kx^2$, moreover, we are immediately assured that

$$y_o = kx_o^2$$

and

$$y_o + \Delta y = k(x_o + \Delta x)^2.$$

Again we subtract the first of these equations from the second and obtain

$$y_o + \Delta y - y_o = k(x_o + \Delta x)^2 - kx_o^2.$$

And again you notice that "y_o" and "$-y_o$" may be dropped from the left-hand side:

$$\Delta y = k(x_o + \Delta x)^2 - kx_o^2 \qquad (18)$$

We now must consider the quantity $(x_o + \Delta x)^2$, which, of course, means $(x_o + \Delta x)(x_o + \Delta x)$; that is, the product of two quantities, one of which is $(x_o + \Delta x)$ and the other of which is also $(x_o + \Delta x)$. In the expression $(x_o + \Delta x)(x_o + \Delta x)$, think of the first parentheses as being a single quantity (which it is!), and the second as the sum of two quantities (which it is!). Then we can apply the distributive principle, saying that

$$(x_o + \Delta x)(x_o + \Delta x) = (x_o + \Delta x)x_o + (x_o + \Delta x)\Delta x.$$

Now we can apply the distributive principle again to both parts on the right:

$$(x_o + \Delta x)^2 = x_o^2 + x_o\Delta x + x_o\Delta x + (\Delta x)^2.$$

Notice that the two terms in the middle on the right are identical and we can collect them, writing:

$$(x_o + \Delta x)^2 = x_o^2 + 2x_o \Delta x + (\Delta x)^2$$

Therefore

$$k(x_o + \Delta x)^2 = kx_o^2 + 2kx_o\Delta x + k(\Delta x)^2$$

applying the distributive principle still another time. We now have another name for "$k(x_o + \Delta x)^2$", which we can put in equation (18). This equation then reads

$$\Delta y = kx_o^2 + 2kx_o\Delta x + k(\Delta x)^2 - kx_o^2.$$

Once again you see that "kx_o^2" and "$-kx_o^2$" can be dropped out, and we have then

$$\Delta y = 2kx_o\Delta x + k(\Delta x)^2.$$

Now multiply both sides of this equation by $\frac{1}{\Delta x}$. We have

$$\frac{1}{\Delta x}\Delta y = \frac{1}{\Delta x} \times 2kx_o\Delta x + \frac{1}{\Delta x} \times k\Delta x \times \Delta x$$

Since multiplying by the reciprocal of a number ($1/\Delta x$) is the same as dividing by the number, we can write this as

$$\frac{\Delta y}{\Delta x} = 2kx_o\frac{\Delta x}{\Delta x} + k\Delta x\frac{\Delta x}{\Delta x}.$$

On the right, you can see that both ($\Delta x/\Delta x$)'s can be dropped out (Do you see why?), and the equation then reads

$$\frac{\Delta y}{\Delta x} = 2kx_o + k\Delta x. \qquad (19)$$

Now remember that we never committed ourselves on how big Δx was, and so equation (19) applies for all Δx, including zero. But if we let Δx be zero, the last term on the right, "$k\Delta x$", is zero, since any number multiplied by zero is zero. Thus the equation becomes

$$\left.\begin{array}{l}\text{The value of } \Delta y/\Delta x \\ \text{when } \Delta x = 0\end{array}\right\} = 2\,kx_o \qquad (20)$$

Notice that this time the value of $\Delta y/\Delta x$ does depend on how big Δx is. Equation (19) applies for all Δx of whatever size; but equation (20) applies only when $\Delta x = 0$. It is the limit of the quantity, $\Delta y/\Delta x$, as Δx becomes zero, that we are most interested in. We cannot call this limit "$\Delta y/\Delta x$" any longer, because its value changes with Δx. In the case of a linear function, it was not necessary to distinguish between "$\Delta y/\Delta x$" and "$\Delta y/\Delta x$ when $\Delta x = 0$", for in that case $\Delta y/\Delta x$ did not depend on the size of Δx. Now we will have to distinguish.

Scientists all over the world use the abbreviation, "dy/dx", to mean "the limit of $\Delta y/\Delta x$ as Δx becomes zero." Notice that dy/dx does not really mean "dy divided by dx." It is not really a fraction -- it only looks like one. Especially you should notice that "dy" does not mean "d times y". The whole thing, dy/dx, is merely a symbol for "the limit of $\Delta y/\Delta x$ as Δx becomes zero." Do not think of it as meaning anything else. The symbol is read "the derivative of y with respect to x", or more simply as "dee-y by dee-x."

The quantity, dy/dx, is called the derivative of y with respect to x. Every time you have a dependent variable, y, which is a function of an independent variable, x, the possibility exists of finding the derivative of y with respect to x. Of course the derivative is not the same for every function. You wouldn't expect it to be, for dy/dx represents the slope of the curve obtained when y is plotted vs x. And different curves may have different slopes, as you know.

For instance, when $y = a + bx$, we have found that dy/dx is simply b. That is, the slope of the curve, $y = a + bx$, is constant. This is something you can see by looking at the curve, which for a linear function, $y = a + bx$, is a straight line. But if $y = kx^2$, the curve is truly curved, and the slope changes depending upon what point of the curve you are talking about. You would therefore expect that the value of dy/dx would change, depending upon what x you are talking about. Equation (20) says that dy/dx is equal to $2kx_o$ whose value clearly depends on what x (that is, x_o) you are talking about. In fact, since x_o in our argument can be any x at all, we might as well drop the subscript from x_o and simply call it "x". Equation (20) then reads:

$$\frac{dy}{dx} = 2kx.$$

Notice that this equation really says that dy/dx is proportional to x. Does this statement agree with your observation that the slope of the curve increases as you go to the right on the graph, $y = kx^2$?

We have found the derivative now for two kinds of functions. They are repeated here for comparison:

$$\begin{aligned} &\text{when } y = a + bx, \qquad && dy/dx = b \\ &\text{when } y = kx^2, \qquad && dy/dx = 2kx. \end{aligned} \qquad (21)$$

9. Uniformly Accelerated Motion

"Uniformly accelerated motion" means simply motion under constant acceleration. We have now covered all that we need for a complete understanding of the relationship between position and time when a body moves under uniform acceleration. The present section will merely gather a few loose ends together.

Let us first recall that you found the ball rolling downhill to have a constant acceleration. This was an experimental finding. You found, furthermore, that defining the word "acceleration" in an acceptable (or agreeable) way leads to the conclusion:

> IF a body travels at constant acceleration starting from rest,
> THEN its velocity is at all times proportional to the time it has been traveling:

$$v = at$$

where a is the acceleration. You verified in your case of a ball starting from rest and rolling downhill that the velocity is in fact proportional to the time and gives a straight line through the origin when plotted against time.

One might now ask: what would the velocity be if the body moved under constant acceleration, but not from rest; i. e., had a non-zero velocity to start with? This problem is very easy.

First we know that, if the acceleration is constant, then by definintion $\Delta v/\Delta t$ (or dv/dt) is constant. This means that the graph of v vs t must be a straight line. (Not necessarily a straight line through the origin: that would mean it started from rest.) This means that v is a linear function of t, and therefore v and t must be related by the equation

$$v = P + Qt$$

where P and Q are constants. Now, can we tell what the constants are? That is, can you give the constants physical meaning? Of course you can! Here is the way you think it out:

The equation holds for any case of uniformly accelerated motion, regardless of what the body's initial (starting) velocity might be. For a particular acceleration and a particular initial velocity, the constants P and Q have particular values. This means that for a particular case, you can calculate v from the equation for any given time, t, at all. If someone gives you t, you can calculate v for him. You can do this because for a particular case, P and Q are given numbers. Now suppose that $t = 0$. The equation then says that $v = P$, since $Q \times t$ is zero when $t=0$. But when $t = 0$, the velocity is the starting velocity, whatever that happens to be. Suppose we call it v_o. Then we know right away that $P = v_o$, and there's one constant that now has physical meaning. We therefore can write our equation with v_o in place of P: $v = v_o + Qt$.

In the equation $v = v_o + Qt$, you now know that $dv/dt = Q$. But dv/dt is by definition the acceleration. Now you know the physical meaning of the other constant: Q is simply the acceleration. Thus we can write

$$v = v_o + at \qquad (22)$$

for the general case. Be sure you understand the meaning of this expression. Here are some questions to help you understand.

If a body starts out with a velocity of 10 cm per second under zero acceleration, what will be its velocity 8 seconds later? If the body is not accelerated, what does common sense tell you the velocity will be at any later time? Does this agree with what the formula tells you?

If abody starts from rest and accelerates 10 cm/sec^2, what will be its velocity 8 seconds later? What is the value of v_o in this case? Does this agree with the equation, $v = at$, which you derived earlier for the case of a body initially at rest?

Do you see why equation (22) is called a "general" formula? It is good even for the cases when v_o and/or a are zero. Suppose a body is initially at rest and is under zero acceleration. What do common sense and the formula tell you is the velocity at a later time?

Are the units of all terms in equation (22) the same, as is required?

If you throw a rock downward from the top of a tall building with an initial velocity of 20 feet per second and the effect of gravity is to accelerate it 32 feet/sec^2, how fast will it be falling after one second, two seconds, three seconds, four seconds. Notice how the velocity increases uniformly (by the same amount) for each additional second of travel.

Let us now return to Experiment 24. Recall that you showed experimentally that the position of a ball rolling downhill is related to the time by the expression

$$p = kt^2.$$

You also have determined the value of k in your experiment. Now notice that, by definition, the velocity for any moving body is dp/dt. When the velocity is constant, the curve, p vs t, is a straight line whose slope, $\Delta p/\Delta t$, is the constant velocity. But when the curve of p vs t is not a straight line, the velocity is not constant. In this case, $\Delta p/\Delta t$ is the slope, not of the curve, but of some secant to the curve. $\Delta p/\Delta t$ then is not the velocity at either of the points where the secant cuts the curve, but at some uncertain point in between. If, however, we allow the two secant intersections to come closer and closer together, $\Delta p/\Delta t$ represents more and more closely the instantaneous velocity. In the limit, when the two intersections have blended into one, $\Delta p/\Delta t$ becomes dp/dt, and this, the slope of the tangent, does represent the velocity.

In Experiment 24, you found dp/dt for your curve, $p = kt^2$, by judging tangents with a ruler. You recognize now that this measurement is inexact in the sense that it is judgement-based because we have no method of drawing a tangent that is unarguably "it." But remember from equation (21) that you learned how to compute the derivative, dy/dx, for any function of the form $y = kx^2$. We therefore now can compute dp/dt for the function $p = kt^2$, and need no longer rely on the uncertain judgement involved in estimating a tangent.

According to equation (21), dp/dt for the function, $p = kt^2$ is simply 2kt. But since dp/dt is also by definition the velocity of the body, we have the interesting conclusion that v = 2kt. This equation says, in agreement with what we previously learned, that the velocity is proportional to the time, with the proportionality constant, 2k. That is:

> IF (as you established experimentally) the position of the body is proportional to the square of the time with the proportionality constant, k, THEN (as you demonstrated logically) the velocity is proportional to the time itself with the proportionality constant 2k.

If the velocity is proportional to the time, however, we have by definition that the proportionality constant is what we call "acceleration." See page 137. Thus 2k is the acceleration of the body rolling downhill. Compute the value of the acceleration in Experiment 24 from the average k you have already determined, and enter this value in the box at the bottom of Table I.

Now finally you can compute the velocity at any time without drawing a tangent. The velocity is always given by $v = v_o + at$, according to equation (22). You can compute v for any given t when you know v_o and a. But in your experiment, v_o was zero and you now know the acceleration, a. Thus you can compute v from the simple expression, v = at. Do so for each t listed in the second column of Table I, using your now-known acceleration. Enter these computed v's in the last column of Table I. Compare them with the "secant velocities" or "average velocities," $\bar{v}$, in column three; and with the "tangent velocities" or "instantaneous velocities," v, in column six. The instantaneous velocity as determined geometrically from tangents should agree quite well with those calculated from the derivative, dp/dt = v = at.

If a body starts from rest and moves under constant acceleration, you have seen that position and time are related by the equation, $p = kt^2$, where k is some constant. Do you see that we have now the same question we had before: is it possible to attach some physical meaning to k? It is possible, for you already know that the acceleration, a, is 2k. This means that k is simply half the acceleration. Thus we can write for a body starting from rest and moving under constant acceleration,

$$p = \frac{1}{2}at^2$$

where a is the acceleration. Now you can calculate the distance traveled by a body under constant acceleration and starting from rest.

For instance, a body falling freely under gravity near the surface of the earth moves with a constant acceleration of 32 ft/sec^2. How far will a body fall in 10 seconds?

Or, try a problem the other way around. How long will it take for a body to fall from the top of the Washington monument, 555 feet to the ground? Taking the origin at the top of the monument, the ground will have the position, p = 555 ft.

Then

$$555 = \frac{1}{2} \times 32 \times t^2 = 16t^2.$$

Divide both sides of the equation by 16. Then

$$34.7 = t^2.$$

Can you solve this equation for t?

Remember that "34.7" and "t^2" are different names for the same thing. If we take the square root of 34.7 we get the same result as when we take the square root of t^2. But what is the square root of t^2? Then

$$t = \sqrt{34.7} \text{ sec}$$

which you can work out yourself. Guess at the answer first. Can you show that the units of t are seconds?

Calculate how far a body falls under gravity in one second, two seconds, three seconds, four seconds, and five seconds. Can you explain the peculiar sequence of results?

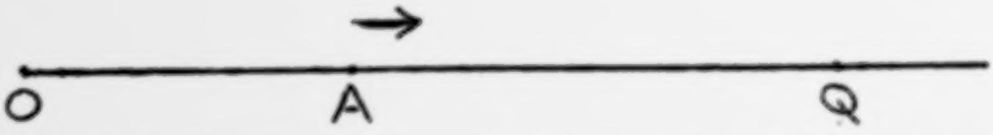
If $p = \frac{1}{2}at^2$, both p and $\frac{1}{2}at^2$ must have the same units. Do they?

10. The Most General Case

You now have seen that a body starting from rest at the origin and moving under uniform acceleration, a, will have a position, p, given by

$$p = \frac{1}{2}at^2 \qquad (23)$$

after traveling t seconds. But do you see that this is a rather narrowly restricted case? It applies only if the body starts at rest and also starts at the origin? Suppose that it starts from rest but instead of starting at the origin, it starts at some other position, p_o. Suppose, for instance, that the body starts



at the point A in the diagram, traveling to the right. Suppose that it starts from this point, from rest, with an acceleration of a. Then the distance it travels to the right from a will be given by equation (23), because there is no reason why we cannot temporarily call A the origin. If in a time, t, the body travels to the point, Q, then we know that

$$AQ = \frac{1}{2}at^2.$$

In this last equation, which holds for all t, suppose that $t = 0$. Then the whole last term, $\frac{1}{2}at^2$, drops out; and we have that, when $t = 0$, $p = \overline{OA}$. But the position at $t = 0$ we have been calling p_o. Hence the term $\overline{OA}$ above is really the physically significant quantity, p_o, the initial position of the body. Thus we have: If a body starts at an initial position, p_o, and at rest, and then moves with constant acceleration, its position at any later time will be given by

$$p = p_o + \frac{1}{2}at^2. \tag{24}$$

Notice that this equation is still restricted by one requirement: that the body start at rest. We have generalized equation (23) to take care of the case when the body starts from a position other than zero. The result is equation (24). Now, can we generalize equation (24) so as to take care of the case when the body starts with a velocity other than zero? We can.

Suppose that you had set up your ball-rolling-downhill experiment in a boxcar. Instead of putting the distance-marks right on the ramp, however, you can see that you could put them on, say, a railing beside the train-track. It might be a little more difficult to make the readings this way, but you can see that the idea would be no different. Then, with the boxcar standing still, you would find that $p = \frac{1}{2}at^2$, just the same as in your experiment. Now suppose that you hold the ball at the top of the ramp, but allow the boxcar to move with uniform velocity along the track. Again you could make readings of the position of the ball by using the marks on the trackside rail. Since the ball stays fixed at the top of the ramp, the only motion it has is the boxcar's motion. which is at constant velocity. The position of the ball would then be given by $p = p_o + vt$, as you found before for motion at constant velocity.

Now think of the two together. If you held the ball at the top of the ramp and the boxcar moved with constant velocity, the position of the ball would be $p_o + vt$. If the boxcar stood still and you released the ball, the position of the ball would be given by $\frac{1}{2}at^2$ farther than it would if it stayed at the top of the ramp. If the ball stays at the top of the ramp, it would be traveling with the same velocity as the boxcar, as measured by your trackside distance-marks.

Suppose now you were in the boxcar, holding the ball at the top of the ramp, and your partner stood at the trackside zero-mark. The boxcar starts a hundred yards down the track, heading toward the zero-mark, moving at constant velocity, v_o. When your partner sees the ball hit the trackside zero-mark, he yells "GO". You can do either of two things:

> You can continue holding the ball. In this case, the position of the ball will be given by $p = p_o + v_o t$; or
>
> You can release the ball. In this case the ball will travel $\frac{1}{2}at^2$ farther than if the ball is not released.

Thus if the ball is released with an initial velocity, and also accelerates, the future position of the ball is given by

$$p = p_o + v_o t + \frac{1}{2}at^2. \qquad (25)$$

There you have it! This is the most general case of uniform acceleration. The formula gives the position of the ball at any future time, t, when it starts at position, p_o, has an initial velocity, v_o, and a constant acceleration, a. Notice the following:

(1) If $a = 0$, there is no acceleration and the body then is traveling at constant velocity. If $a = 0$, does equation (25) become identical with equation (13), the one developed for motion at constant velocity?

(2) If the body starts from rest and moves under constant acceleration, what is the value of v_o in equation (25)? Does this equation then become identical with equation (24)?

(3) What does it mean if v_o and a are both zero? Does equation (25) give a sensible result when v_o and a are zero?

(4) Do all terms in equation (25) have the same units?

(5) If you subtract p_o from both sides of equation (25), you get

$$p - p_o = v_o t + \frac{1}{2}at^2.$$

What is the meaning of the left-hand side of this equation?

Now try your hand at a problem: A boy throws a rock downward from the top of a fall building. If the rock accelerates downward by gravity at 32 ft/sec^2 and he throws it with an initial velocity of 40 ft/sec, how far will it have fallen in 10 seconds? How far would it have fallen if he had merely dropped the rock without throwing it? Do you see how little effect an initial downward throw has, if the body travels for a relatively long time? Try making the same comparison if the rock travels for only one second.

Equation (25) is extremely important in dealing with the behavior of missiles and rockets.

Experiment 1

Measuring Lengths with a Ruler

In this very easy experiment, you will measure the lengths of a number of plastic rods, using a ruler graduated in centimeters and tenths of a centimeter. Tenths of a centimeter are also called "millimeters". The sticks will also be measured by several other people in your class. After everyone is finished, you will be able to compare your measurements with those of others who measured the same sticks.

Before you start this experiment, your teacher will explain to you how to make the measurements and how to record them. It is just as important to record measurements properly as it is to make them properly. Be sure you understand what to do before you start the experiment. Also, read Sections 3 and 4 in your textbook before you begin. You should understand about Making Measurements and Significant Figures before you start.

In this experiment, as well as in all others, be very careful with all the apparatus you use. Do not damage the sticks or the rulers. Do not make any marks on them. Be very careful not to drop pieces of apparatus. Be careful that the edges of the rulers and sticks are not bumped so that they become dented or mashed.

Procedure: Your teacher will supply you with six sticks of different lengths. Measure each one this way:

Lay one end of the stick so that it lies as nearly as you can judge on the zero-centimeter mark of the ruler. On some rulers there is a zero-centimeter mark actually appearing on the ruler; on others, the zero-mark is simply the very end of the ruler. Examine yours and decide which type you have. Lay the stick so that it lies along the ruler. Look at the other end of the stick and decide which tenth-of-a-centimeter mark on the ruler the other end lies closest to. Select this mark as representing the length of the ruler. Read it, and record the length before you forget it. Make your record in the table on the data sheet (next page). Be sure you record also the number of the stick. The last column in the table is for entering the average length of each stick for everyone in the class who measured it. Do not compare your measurements with anyone else's until everyone is finished.

When everyone is finished measuring, your teacher will call for the results obtained by each different person who measured each stick and will write the different results on the board. If the measured lengths for any stick are not all alike, find the average, remembering the business about significant figures. Enter the averages obtained for the sticks you measured in the last column of the table.

Experiment 1
Data Sheet

Table I. Measured Length of Some Sticks

Stick No.	Measured Length	Average of Several Measmts

Enter your measurements from Experiment 1 in the table above. Record your observations as you make them, and don't forget to include the units. Wait until the whole class is finished before you compute the averages in the last column of the table.

Experiment 2
The Adding of Measured Lengths

In this experiment you will take some sticks whose lengths have already been measured, lay them down end to end, and then measure their combined lengths. We will then see whether the measurement you make of the combined length could somehow have been predicted from knowing the individual lengths.

Can you already make such a prediction? Suppose you have a straigh stick four feet long and another one three feet long. Lay them down end-to-end so that they are exactly in line with one end of one stick butted tightly against one end of the other. How far will it be from the free end of the first stick to the free end of the second? You would answer "seven feet", wouldn't you? Where did you get this prediction of seven feet?

You probably would answer something like this: "If I have four apples in one pile and three apples in another and then shove the two piles together into one pile, I know that the total apples in the big pile is the sum of those in the two piles. The same is true whether the objects in the pile are apples, or pigs, or teacups, or cuckoo-clocks. Why shouldn't it be true of feet as well?"

But there is something very doubtful about this argument. It is true that combining a pile of three objects with a pile of four objects gives a pile of seven objects, no matter what kind of individual objects you talk about. In fact, it is true not only of three and four, but of all other natural numbers as well. If there are A objects in one pile and B objects in the other pile, we say that the number of objects in the combined pile is "A + B". Here, remember that A and B are numbers. The symbol "+" and the word "sum" are defined in such a way that the sum, A + B, is the number of objects in the combined pile. You have learned to add numbers in a way that makes this always true. But what right have you to think of the "4" in the quantity "4 feet" as meaning a pile of four feet that can be lumped together with a pile of three feet to make a pile of seven feet? The fact that the method of adding that you learned in arithmetic gives the right answer when used on numbers by no means gives you the right to say that it will also work with quantities that are not purely numbers. On the other hand, we don't have the right to say that it won't work, either! We just don't know. Let's try it!

Procedure: Obtain three sticks whose lengths have already been measured to the nearest tenth of a centimeter. Lay them end-to-end in a straight line. Push them gently against the edge of the ruler to make sure they are in line. Be sure they are butted tightly against each other. Then measure the distance with your ruler from one free end of the train to the other. Measure the total length the same way you did in Experiment 1. Enter the result in the seventh column of the table below. Remember to put down the units.

Before disturbing the sticks, write the number marked on your left hand stick in the first column of the table under the heading "No."; write the number marked on your middle stick in the third column; and write the number of your right hand stick in the fifth column. Do not write anything now in the second, fourth, and sixth columns, under the heading "Length."

Now rearrange the sticks in a different order and measure the overall length again. Record your result in the seventh column of the table and also record the numbers of the left, middle, and right sticks in columns 1, 3, and 5. Obtain three new sticks from your teacher and repeat the work, entering the results again in the proper columns. Make two measurements of the combined length with the sticks in two different orders. Then repeat the whole thing (two measurements) with a new set of three sticks.

After you have finished the measurement, ask your teacher to tell you the known lengths of the sticks. These you can get by giving the number of each stick (you recorded these numbers in the table) and having your teacher tell you the known length of the stick with that number marked on it. Enter these quantities in their proper places in the second, fourth, and sixth columns. Don't forget the units.

Your table now has six lines of data, complete except for the last column. You obtain the last column by adding the individual lengths of the left, middle, and right sticks. Do this for each line.

Experiment 2
Data Sheet

Table I. Measured Total Length of Combined Sticks

	Left Stick		Middle Stick		Right Stick		Total Length	
	No.	Length	No.	Length	No.	Length	Measured	Computed
First set of sticks								
Second set of sticks								
Third set of sticks								

Record your observations from Experiment 2 in this table, as you make them. When you are ready to measure your first line-up of sticks, first record the sticks' numbers in the first, third, and fifth columns under "no." Do not record anything yet in columns 2, 4, and 6. Then measure the total length and record it in the seventh column under "Measured." Don't forget the units. Do not bother to measure individual sticks.

Next, rearrange the sticks in a different order, recording the sticks' numbers in the second line of the table. Then measure the combined length and record it in column 7.

Repeat the whole thing, using two other sets of sticks.

Find out from your teacher the known lengths of the sticks you used, and record them (units!) in the proper places. Then compute the total length for each set, entering the sum in the last column.

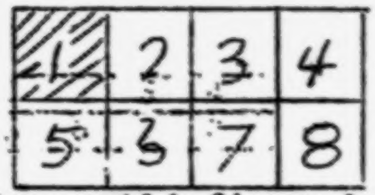
Experiment 3
Measuring Areas

In this experiment you will use what you have learned about significant figures and about measuring lengths to measure the areas of some plastic cards which your teacher will give you. You will have to remember some rules about how to compute the areas of rectangles, triangles, and circles. You will also have to use your ingenuity.

Your teacher will first give you three cards: one rectangle, one circle, and one triangle. You are to make the necessary measurements on these cards that will permit you to computeetheir areas. Be very careful with the cards. Do not bend or fold them nor allow the edges to become damaged. Do the rectangle first, then the circle, then the triangle, as described below.

How might you measure the area of a rectangular card? You do not have an "area-measurer" that will measure areas as a ruler measures lengths. What then can you do? Well, you remember that area is really a derived quantity, the unit of which is the area covered by a square that measures 1 cm on each side. You could make for yourself a little "unit area", which might be a tiny square card measuring 1 cm each way. Te measure the area of a given card you could "lay off" the unit over the card to be measured, seeing how many times were needed to cover it without overlap. You would have troubles fitting the measuring card (the little unit square) to the big card that you are measuring if the card were irregular in shape or if the laying off did not come out even, but there is no problem otherwise.

For instance, suppose the card you wanted to measure looked like the one in this picture. You could first lay off the shaded area with your unit measure, then block number two, then three, etc., with no overlap ever and no space left over. If the rectangle were exactly 4 units long and 2 units wide, the job could be done as in the picture with everything coming out even. Since you now have two rows of four unit squares, you have 2 times 4 unit squares, or 8 units needed to cover the rectangle. We say that the area of the rectangle is 8 square cm. You will notice that the reason we could fit four unit squares in the length of the rectangle is simply that the rectangle is 4 cm long. The reason we could fit exactly two unit squares in the width is that the rectangle is exactly 2 cm wide. Thus when we multiplied 2 x 4 to get the number of unit squares, we were also multiplying 2 cm by 4 cm to get 8 square centimeters.



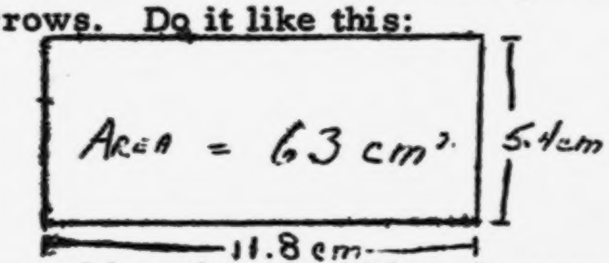
There are several matters that would have to be examined more carefully, however. First, does it make any difference in what order you lay off the unit squares? Second, is it always true that the number of square centimeters in a rectangle may be obtained by multiplying the number of centimeters in the length by the number of centimeters in the width? Does it matter whether you multiply width times length or length times width? What do you do if the number of times you lay off the unit square along the length does not come out even? We learn, partly by experiment and partly by logical thinking, that the area of a rectangle is always given by multiplying the number of length units in the length of the rectangle by the number of units in the width. The product of these two numbers will be the number of square units in the area, no matter what units are used to measure both length and width (as long as they are measured in the same units). Proving that "Area equals Length times Width" requires a very careful examination of principles of geometry. We will simply accept these results here and not attempt to prove it beyond using the 2 by 4 diagrams of blocks above.

When the area is not rectangular, then what do you do? If the shape is simple enough, geometry can answer this question, too. You probably already have met formulas for the area of a circle and for the area of a triangle. We will use these formulas too without further proof. To help you in case you have forgotten, here they are:

For a rectangle, area = length x width
For a triangle, area = half of base x altitude
For a circle, area = pi x radius x radius

where pi is 3.1416, to five significant figures.

Procedure: Take the rectangular card and measure both its length and its width to the nearest tenth of a centimeter. Make a neat sketch of your rectangle to scale in the box labeled "Rectangle" on the data sheet. Draw arrows to show the dimensions, writing the length and width that you measured in the gaps in the arrows. Do it like this:



After you have measured length and width and recorded them on the sketch, use the formula to compute the area of your rectangle and record the area inside the sketch as in the above sample. Don't forget the rule about significant figures and remember to put in the units.

Next take the circular card. What information do you need to compute the area? You need the radius, don't you? Now the radius of a circle is the distance from the center of a circle to its boundary, isn't it? If the center of your circle is not marked, how can you measure the radius? The radius is half the diameter, and the diameter is the greatest distance through the circle. So, put the zero end of your ruler on the edge of the circle and be sure to keep it there. Then point the other end of the ruler so that the edge of the ruler passes through the point where you think the center is. Move the ruler back and forth a little, being sure to keet the zero mark of the ruler on the edge of the circle. Now watch where the opposite edge of the circle falls on the ruler. As you move the ruler back and forth, keeping the zero-mark on the edge, the opposite edge of the circle will fall at different places on the ruler. The largest reading you can get is the diameter. Half of this is the radius. Read the diameter to the nearest 0.1 centimeter.

Draw a circle in the box labeled "Circle" on the data sheet. Inside the circle, write neatly, "Diameter = _____", filling in the blank with the diameter you measured. Then compute the area using the formula, Area $= \pi r^2$. If $\pi = 3.1416$, how many of these five significant figures should you use to compute the area? Write the computed area inside the circle as you did for the rectangle.

Next take the triangular card. To compute its area, you need the base and the altitude. Any side of the triangle may be called its base; it doesn't matter which one. Select any side you please as base and place the triangle flat on the table in front of you with your chosen base nearest you. Now what is the altitude of this triangle? Once a side is selected as base, the altitude of the triangle is the perpendicular distance from the opposite corner of the triangle to the base. Hold your ruler so that the zero mark lies at this corner, the rest of the ruler pointing toward you across the base. Keeping the zero mark on the corner, waggle the ruler back and forth until you judge the ruler to be perpendicular to the base. You can use a corner of your rectangular card to help you judge the right angle. Now read the distance from corner to base. This is the altitude; read it to the nearest 0.1 centimeter.

In the box on the data sheet labeled "Triangle, first base", make a neat scale drawing of your triangle, your chosen base at the bottom. (It isn't so easy to make a scale drawing of an irregular triangle without drawing instruments. Your teacher will show you how to do it using the lines drawn across the corners of your plastic triangular card.) Using arrows to show dimensions as you did for the rectangle, show the base and altitude for this triangle. Then using the formula for the area of a triangle, compute the area of yours and record it inside the sketch of the triangle. Remember about units and significant figures.

Now choose another side for the base of the triangle and repeat the measurements of base and altitude for this base. Make a scale drawing in the box labeled "Triangle, second base", putting in dimensions as before. Compute the area again, using the new base and altitude. Then repeat the whole thing once more, using the third side of the triangle as base.

If you still have time, your teacher may want you to measure the areas of some more complicated shapes. You will have to use your ingenuity in deciding what to measure and how to compute the areas of these cards. You may have this hint: all the shapes are made up of rectangles, triangles, and circles or parts of circles. Try to discover for yourself how to measure their areas. Make sketches and show their dimensions in the unlabeled boxes.

Experiment 3
Data Sheet

Make neat scale drawings of your cards in the boxes below

Rectangle #

Circle #

Triangle

First Base

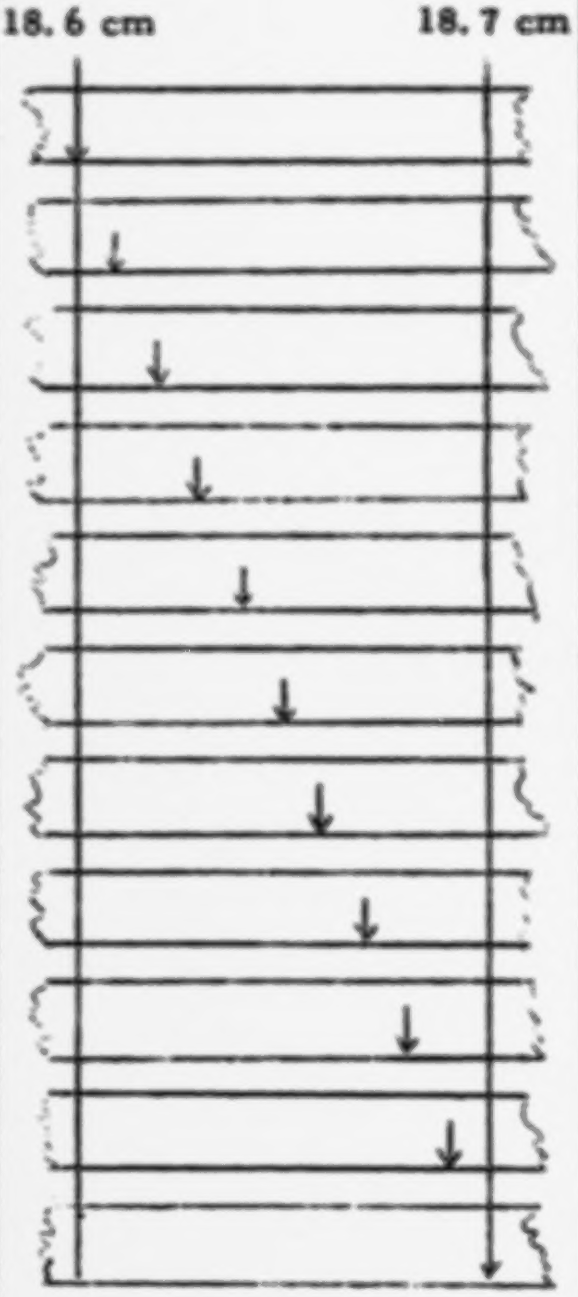
Second Base

Third Base

If you measure any other cards, make scale drawings of it (or them) in the space below.

Experiment 4
Decimal Estimation

This experiment is intended to give you practice and confidence in reading a scale by decimal estimation. First obtain a few plastic rods and a centimeter scale from your teacher. Practice measuring these rods by decimal estimation. Measure them exactly as you did in Experiment 1, but instead of calling the length according to the nearest graduate mark on the ruler, do your best to estimate how far between the graduations it lies. The diagram below may be helpful. The horizontal line is the edge of the ruler; the two vertical marks are the 6th and 7th marks between the 18 cm and 19 cm marks; the arrow represents the end of the thing you are measuring.



If as nearly as you can tell, the end of the rod lies	where the arrow is here,	record the length as:
Right on the six-tenths mark----		---18. 60 cm
Just barely past the mark------		---18. 61 cm
A little more past the mark----		---18. 62 cm
Amount from 6 to arrow about half as much as arrow to 7-		---18. 63 cm
A little less than halfway-----		---18. 64 cm
Right in the middle----------		---18. 65 cm
A little beyond the middle-----		---18. 66 cm
Amount from 6 to arrow about twice as much as arrow to 7-		---18. 67 cm
Not quite as much as the one next below ---------------		---18. 68 cm
Just barely short of the 7 mark-		---18. 69 cm
Right on the 7 mark------------		---18. 70 cm

After you have practiced making a few measurements this way (maybe ten or so), begin making the measurements you will record. Measure five different rods. Record the rod number and your measured length in the table on the data sheet. Each length should be recorded to the nearest 0.01 cm, the last figure being estimated.

At least six, preferably more, people should measure each rod. Rods will be passed around so that each student measures five different ones, and each rod is measured by at least six different students. Do not ask anyone else "What did you get?" in order to compare your measurement with his. Wait until everyone has made all his measurements before any comparing. Then the whole class will compare together.

The teacher will now ask each student who measured rod #1 to call out the length he recorded, and will write each value on the board. The same will be done for every rod that anyone in the class measured. When all the measurements for any one rod are listed, they should be averaged. The average length should be recorded in the data table. Don't forget about significant figures: the average should be rounded off to the nearest 0.01 cm, just like the individual measurements.

Record the average length for each stick you measured in the third column of the data table. Was your measurement the same as the average? Probably not. Compare your result with the average by subtracting whichever is smaller from whichever is larger. This difference is called a deviation. Record the deviations from the average in the last column of the data table. If your measurement agreed exactly with the average, you would record your deviation at "0.00 cm." If your measurement was, say 0.02 cm more than the average, you record the deviation as "+0.02 cm." If your measurement was 0.03 cm less than the average, record your deviation as "-0.03 cm." Your deviation is minus if your measurement is less than the average; plus, if your measurement is greater than the average.

Experiment 4
Data Sheet

Table I. Measured Lengths of Some Rods

Number of Rod	Measured Length	Average of several Measured Lengths	Your Deviation

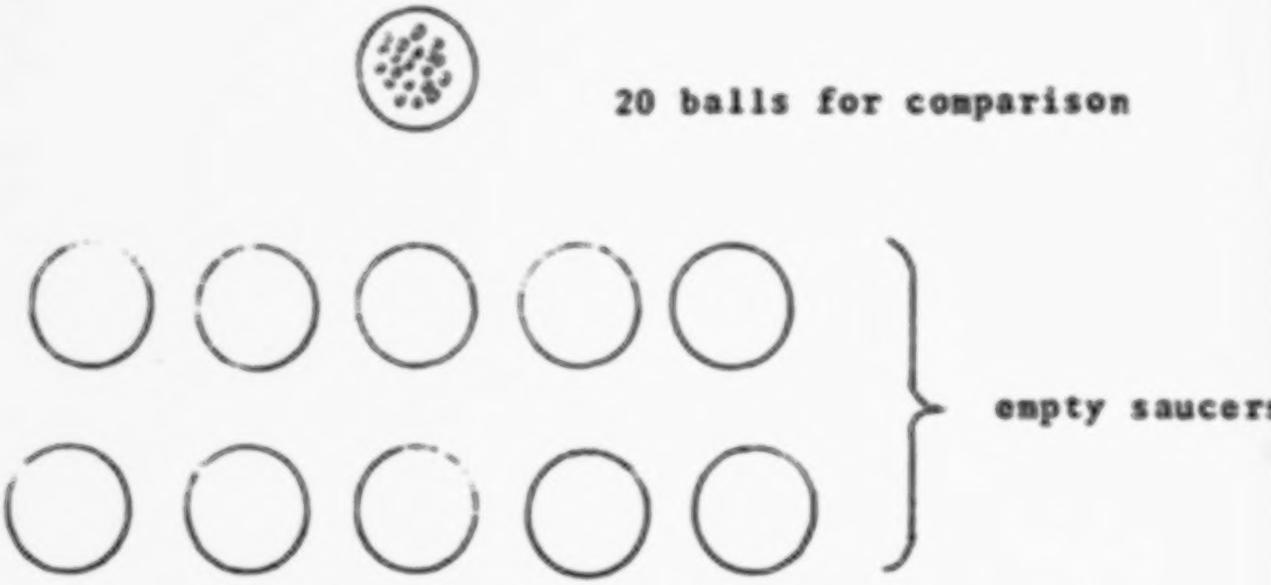
Enter the number of each rod you measure in the first column and its length as you measure it to the nearest 0.01 cm in the second column. Measure five rods (more if you have time).

After everyone is finished, enter the average of all measurements made on each of your rods in column three. Subtract column three from column two to get the deviation. Record the deviation in the last column, + if your measurement is greater than the average and - if your measurement is less than the average.

Keep these data because you will need them again for Experiment 5.

Experiment 5
Averages and Deviations

This experiment is a kind of guessing game in which you can find out how good a guesser you or some of your classmates are. Your teacher will have at hand a jar full of some kind of small uniform objects like ball-bearings, marbles, or dried beans. There will also be eleven small saucers arranged on a table. They should be arranged something like this:



One of the students should be elected to do the guessing.

Someone should count out exactly 20 balls (or beans or marbles or whatever) and place them in the top saucer. The guesser takes the jar of balls and pours into each saucer what looks to him like 20 balls, looking at the top saucer for reference as often as he wishes. He must portion out an estimated 20 balls to each dish in a time limit of two minutes - ten dishes in all. After having filled all the dishes, he may use any time remaining to add to any dish that seems to him to be short of 20, or to remove from any dish any balls that seem to him too many. He may not count the balls at any time. He may adjust any dish's portion by adding or removing balls and comparing with the reference dish, but he may not count the balls.

After the two-minute limit has passed, the balls in each dish should be counted and listed in the first empty column of Table I on the data sheet. The balls are then returned to the jar (except those in the reference dish), and the whole game repeated with another student as guesser. As many guessers should play the game as time permits. The name of each guesser is entered in one of the boxes just below the top double line of Table I, at the top of his column of guesses.

Each column in Table I is now averaged to find the average number that each player guessed for 20. When you average ten or more numbers that are close together, it is usually agreed that you are entitled to one more significant figure in the average than there are in the numbers being averaged. It is perfectly legitimate, therefore, to enter the average as "19.7," or "21.3" or whatever it comes to. That is, carry to one decimal place the averages listed in the boxes below the bottom double line of the table.

After you have computed the average for a player, compute the deviation from the average that he made on each estimate. Then compute his average deviation and write it in the bottom box of his column. Do this for each player.

You should now return to the results of Experiment 4. There, a number of different people made measurements of the length of a certain rod. In Table I of Experiment 4, you listed your own measurement of the length of, say, rod #1. You also listed the average of the length-measurements of this rod as obtained by several people, and the deviation of your own measurement from this average. We will now treat these data in much the same way as in the guessing game. Use Table II on the second data sheet for this experiment.

The teacher will call for, say, the results obtained by all students who measured rod #1 and write these measurements on the board. If you measured rod #1, copy all these measurements into the first column of boxes in the table, writing the number of the rod at the head of the column. Do the same in other columns of the table for the other rods you measured.

In the box first below the lower double line in the table, write the average of the measurements in the column. Also compute the deviation of each individual measurement from the average and then calculate the average deviations, writing this value in the second box below the double line.

Experiment 5
Data Sheet #1

Table I. Estimation of Numbers

Number of balls guessed by

Dish No.							
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							
Average							
Average Deviation							

Write in the box at the top of each column the name of a guesser and in the column below his name, the actual number of balls he guessed for each dish. Calculate his average guess and enter that in his column in the box on the line labeled "Average." In the box below that, enter the average deviation of his guesses. Do the same for each guesser.

Experiment 5
Data Sheet #2

The data on this page come from Experiment 4.

Table II. Measurements of Rod Lengths

	Measured Lengths of Rod Number							
No. of Rod →								
List here the results obtained by all students measuring the rod ←								
Average Length								
Average Deviation								

The form of this Table is much like that of Table I. In the top box of each column, write the number of a rod that you measured in Experiment 4. In the boxes beneath this number, write the lengths of this rod as measured by all the other people who measured it. Write the average of these in the box second from the bottom and the average deviation in the bottom box.

Experiment 6
Determining $\sqrt{2}$ by Measurement

In this experiment you will make some measurements that will allow you to find the value of a very important absolute constant. This constant is a number (without units), yet you would never meet it if the only numbers you ever met were the numbers used in counting, or the numbers you get by adding, multiplying, dividing, or subtracting the numbers used in counting. It therefore has to be determined by measuring or by some other peculiear way. We will measure it.

Procedure: Your teacher will supply you with several metal or plastic squares of different sizes. Take one of these squares and measure very carefully the length of its edge. Measure to the nearest 0.01 cm and record the edge-length in Table I. Also measure the diagonal of the same square to the nearest 0.01 cm and record it in the table. When you measure the diagonal, be sure you measure from the very point of one corner to the very point of the opposite one. If the points have been damaged by mashing, you cannot use that square. Make the same measurements for at least six squares of different sizes, entering the edge and diagonal that you measure for each square on a different line of the table.

After you have made all your measurements, put the squares away and work out the last column of the table. Do this by taking the measured value of the diagonal of your first square and dividing it by the measured value of the edge. The quotient (or ratio) should be entered on the first line in the last column. Then calculate the ratio for each line in the table. How many significant figures are you entitled to in the ratios? Is there anything peculiar that you notice about the ratio, "diagonal divided by edge," for different r

Experiment 6
Data Sheet

Table I. Measured Values of the Edge-Lengths and Diagonals of Some Squares

Square No.	Edge of Square, cm	Diagonal cm	Ratio = $\frac{\text{Diag.}}{\text{Edge}}$
Average of Observed Ratios			

Computed Value of $\sqrt{2}$	

Enter your measurements from Experiment 6 in the table above, first t three columns. After you have finished measuring, work out the ratio "diagonal divided by edge" for each line of the table and record the ratio in the last column. Find the average of these ratios and record the average, too.

Finally, work out $\sqrt{2}$, correct to five significant figures, and place this value in the box at the lower right.

Experiment 7
Calculating $\sqrt{2}$ by Continued Fractions

A fraction like the big complicated one on the next page is called a "continued fraction." The thing that makes it a continued (instead of an ordinary) fraction is that the denominator is a fraction whose denominator is itself a fraction whose denominator is itself a fraction whose denominator is itself a fraction, and so on and on. Of course, you can't spend forever going "on and on," but you can get as close as you please to the right result if you go far enough. You will get $\sqrt{2}$ correct to five decimal places by taking seven 2's in the big fraction as written on the worksheet.

The secret of harnessing this fraction is to start at the very end. Notice the innermost circle drawn around $\frac{1}{2\frac{1}{2}}$. Since $2\frac{1}{2}$ is 2. 5, you can easily work out $1/2.5$ and show that it equals 0. 40000. There fore the innermost circled fraction is 0. 40000. This is already marked for you by the horizontal line pointing to the innermost circle.

The next innermost circle then really says $\frac{1}{2+0.40000}$ Work out $1/2.4$ to five decimal places. You should get 0. 41667. What will you then write at the second horizontal line for the value of the fraction in the next innermost circle? Write it.

Then the third innermost fraction is $\frac{1}{2+0.41667}$ which you can work out by dividing 2. 41667 into 1. This result you put on the third horizontal line. Now you should be able to finish it by yourself. Write each successive partial result on successive horizontal lines.

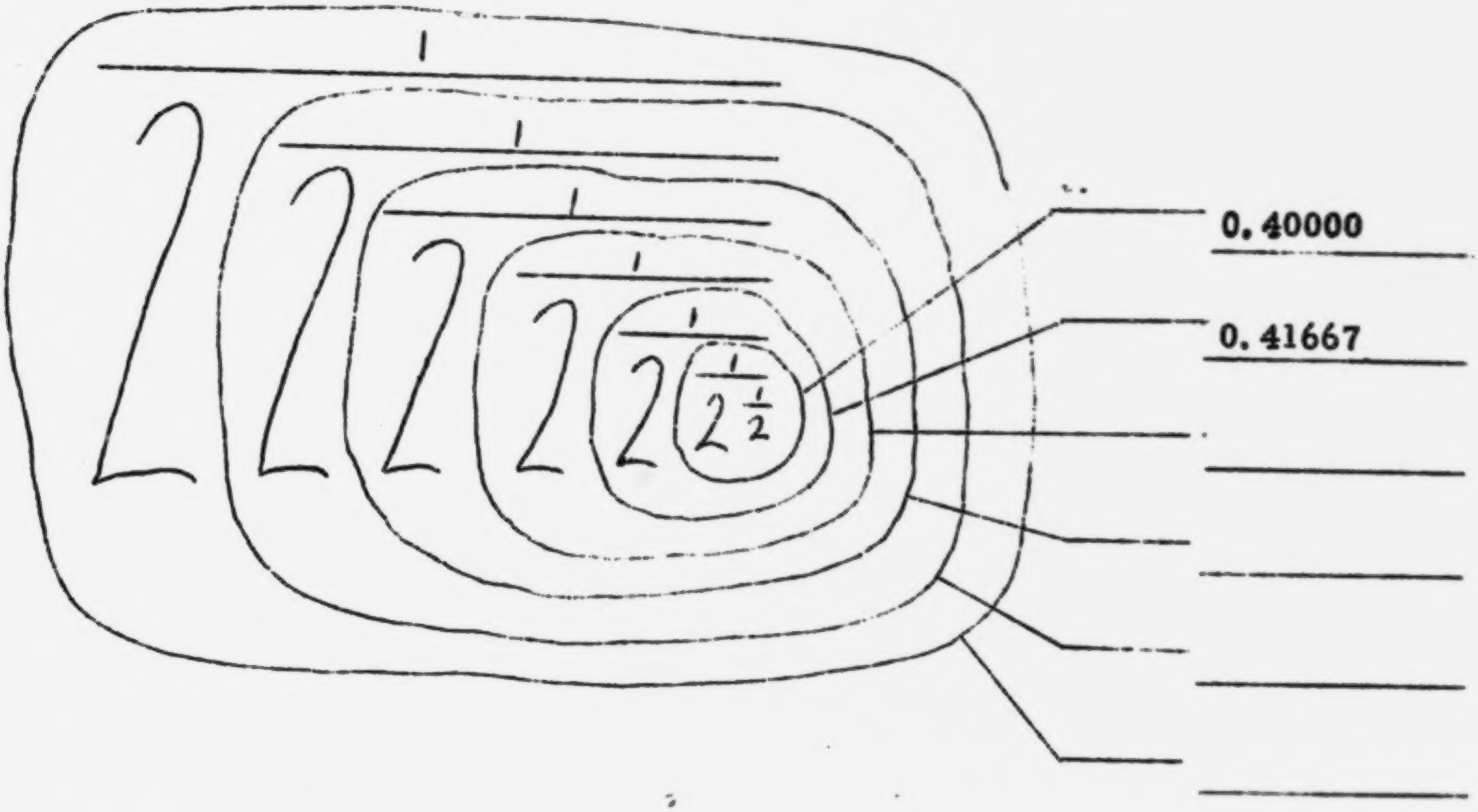
Each time you work out the value of one of these fractions as you go along, you could stop and add the result to 1 and the sum would then be approximately $\sqrt{2}$. But the more fractions you include, the closer you get to exactly $\sqrt{2}$, though you never get it exactly.

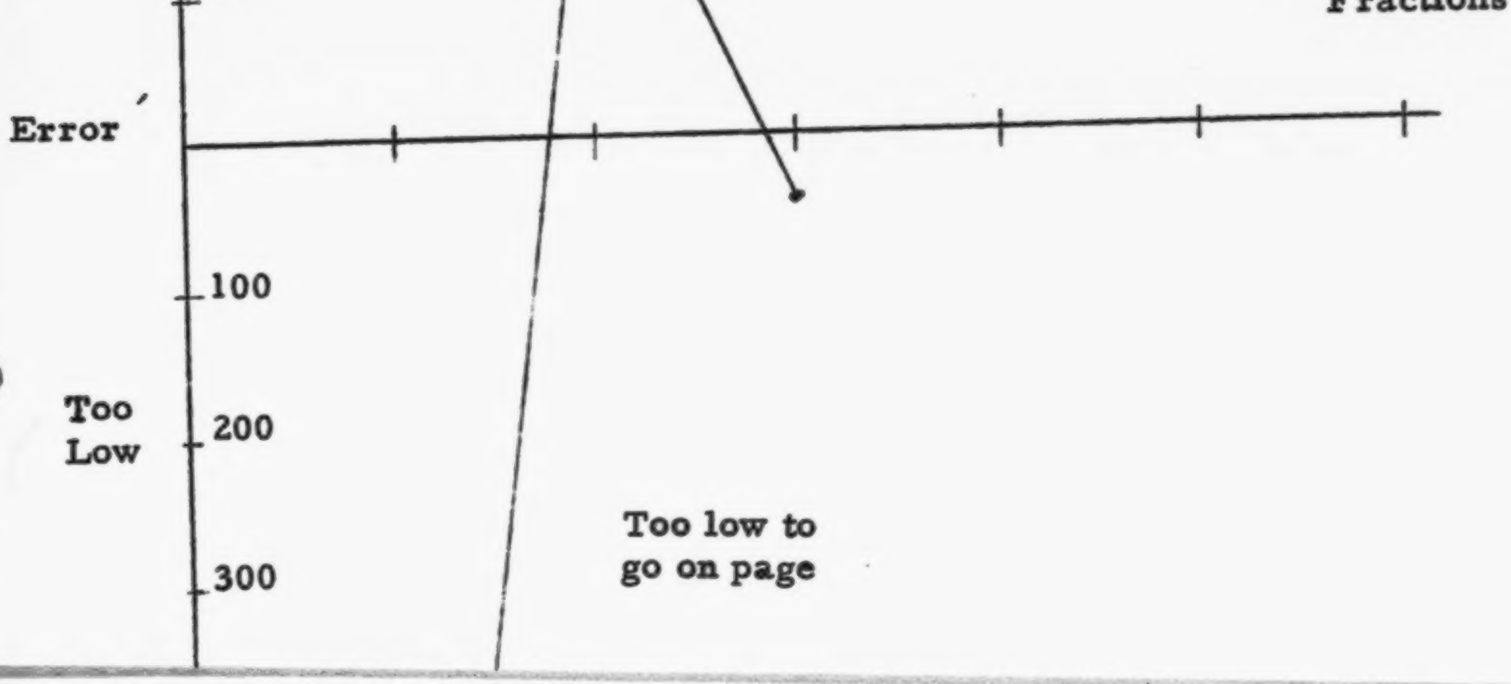
The correct value of $\sqrt{2}$ is 1. 41421 to five decimal places. Notice that the very first fraction would have given you 1. 40000, correct to one decimal place but too small. The second fraction would have given you 1. 41667, correct to two decimal places, but too large by 256 units in the fifth decimal place. The next fraction you will find is too small by a smaller error and the next too large by a still smaller error. The results, in other words swing back and forth past the "truth," but get steadily closer. For six fractions, the error is only one in the fifth decimal place.

You might find it interesting to make a graph showing how the successive results zero-in toward the right value. The one on the worksheet is started for you. You finish it. On this graph you plot the errors made by stopping the calculation after only one fraction, two fractions, three fractions, etc. If the calculated value is too high (i. e., greater than 1. 41421), plot the error upward; if too low, plot it downward. The "error" is calculated by finding the difference between your calculated value to five decimal places for $\sqrt{2}$. and its actual value of 1. 41421, dropping the decimal point.

Experiment 7
Data Sheet

Successive Values of the Continued Fraction


1
2
1
2
1
2
1
2
1
2
1
2 1/2
0.40000
0.41667



Experiment 8
Measurement of the Constant, π

The purpose of this experiment is to determine by measurement the value of a very important constant called π. The constant may be defined as the ratio, circumference divided by the diameter, of a circle. What does this definition suggest that you do in order to measure the value of π ? Let's do it.

Procedure: Your teacher will furnish you with some metal or plastic circular discs of various sizes. You are to measure the diameter and the circumference of each circle. In Experiment 3, you learned one way to measure the diameter of a circle, but this method is not good enough for the present experiment. Your teacher will demonstrate to you a better method -- the "caliper" method -- for measuring the diameter of a circle, Take one circle and measure its diameter to the nearest 0.01 cm. Enter the value in Table I on the data sheet.

To measure the circumference of the disc, get a narrow strip of thin strong paper and an ordinary pin. Wrap the strip of paper carefully -- straight and tight -- around the edge of the disc so that the paper makes a kind of raised rim around the disc. Be sure that the paper overlaps itself a little so that there is a region where the paper is double thick. Hold the disc with the paper drawn very tight and prick the paper with a pin somewhere through the double thickness. Both thicknesses must be pricked. Now unwrap the paper strip and pencil a little circle around each of the two pin pricks. (The only purpose of the penciled circle is to assure that you don't lose sight of the pin pricks.)

Do you see that the distance between pinpricks after the paper strip is stretched out equals the circumference of the disc? Measure the distance between pin pricks to the nearest 0.01 cm and record the measurement in Table I. Cross out the two penciled circles on the paper strip to make sure you don't confuse them with later measurements and then repeat the whole process with other discs. Measure at least six discs this way, recording diameters and circumferences in the table, each disc on a different line of the table.

Next you are to repeat the measurements you made above but this time make the measurements in units other than metric units (centimeters). First, select any one of the discs you measured and recorded in Table I and prepare to make the measurements again. There is little point in repeating the measurements in centimeters, however, so simply copy on the first line of Table II the data you recorded for this disc in Table I. Then repeat the measurements of diameter and circumference on the same disc using a ruler graduated in inches. Record these measurements in the second line of Table II. Make this and the next measurement on only one (the same) disc.

Finally, make the measurements using a ruler of your own manufacture. Take a small piece of paper with a straight edge and make two perpendicular marks at the edge like the two marks at the edge of the page after the end of the line you are now reading. It doesn't really matter how far apart they are, but make them about the width of one of your fingers. This will be your unit of measurement; since it is not an inch or a centimeter, you will have to make up your own name for it -- say "widget." Write the name you give the unit in the third line of Table II, first column under "Measured in." Now make a ruler graduated in widgets (or whatever name you choose). Do this by taking a strip of cardboard about one foot long and transfer to the edge of the strip, time after time, marks that are exactly one widget apart, using the original widget you marked off on the small piece of paper. Use this ruler for measuring the diameter and circumference again of the same disc, recording these data on the third line of Table II. Your teacher will show you how to estimate fractions of a widget.

Now work out the ratio, circumference divided by diameter, for this disc for the three units of measurement you used, recording the ratios in the last column of Table II.

Experiment 8
Data Sheet

Table I
Measurement of the Constant, π

No. of Disc	Diameter cm	Circumference cm	$\frac{\text{Circumference}}{\text{Diameter}}$
	Average		

Enter in the table above your measurements of the diameters and circumferences for the six discs you measured. Divided each circumference by its diameter and record the ratio in the last column. Calculate the average of your six ratios and enter the average in the bottom box.

Table II
Circumference/Diameter Ratio in Non-Metric Units

Measured in	Circumference	Diameter	Ratio
Centimeters			
Inches			

Fill out this table in the same way as Table I. The first line will be the same as some line of Table I. The second line should have the same measurements made in inches, and the third line in some other unit of your own invention.

Experiment 9
Calculation of π

From $\pi = 4\left[1/2 + 1/3 - 1/15 + 1/35 - 1/63 + \text{etc.}\right]$

The denominators of the fractions after 1/2 in the parentheses are each "one less than the square of the even numbers in order." In the first column below, fill in the even numbers up to 32. In the second column, write the squares of these numbers. In the third column, write one less than the squares, and in the fourth column write the fraction having that numbers as denominator and 1 as numerator. Work out the decimal value of each of these fractions by dividing the denominator into 1 to four decimal places, and write this decimal number in the fifth column. In the next column tell whether this term is to be added or subtracted. Look at the series at the top of the page and you will see that the second fraction is to be added, the third subtracted, and they alternate +-+- ever after.

In the next to last column, write the partial sums. Start with 0. 5000, add the 0. 3333 to it to get 0. 8333. Next you subtract the 0. 0667 appearing in column 5 from 0. 8333 and write the difference, 0. 7666, in the partial sum column under 0. 8333. Finally, multiply the partial sum by 4 to get an approximation to π .

Some of the numbers are already filled in to get you started. When you are finished, notice how the numbers in the last column swing back and forth around π , always getting closer.

Experiment 9
Worksheet

Calculation of π

Even Nos.	Their Squares	One Less	$\frac{1}{\text{One Less}}$	Decimal Value	+ or -	Partial Sum	4 x Sum
First term = 1/2 →						0.5000	2.0000
2	4	3	1/3	0.3333	+	0.8333	3.3332
4	16	15	1/15	0.0667	-	0.7666	3.0664
6	36	35	1/35		+		
32	1024	1023	1/1023	0.0010	-		

Experiment 10

Height and Distance along a Ramp

As you walk up a ramp, you know that the height you stand above ground level at any moment is a function of how far along the ramp you have walked. T The purpose of the present experiment is to help solidify in your mind the idea of numerical relationships existing between two quantities one of which is a function of the other. By carrying out the experiment with two ramps of different character, we will try to distinguish between "known function" and "unknown function."

Procedure: Your teacher will supply you with two plastic "ramps" -- really strips of plastic -- one straight and one crooked. Unlike the kind of ramp you usually see, however, the "walk-on" part of these ramps is to be regarded as the edge of the strip, not the flat side. (This makes the ramp a little troublesome to walk on, but you are not going to walk on it anyway.) By setting up the strip so that it is inclined to the table top, you have a model of a real ramp on which you can make some measurements. You will also be supplied with a breadboard, a dowel post, a clothespin clamp, a ruler, a protractor, and a dowel pin.

Your teacher will set up a sample apparatus to show you how to build your own. Use the straight ramp first. Everyone in the class should have his ramp inclined at a different angle, ranging from a gentle slope of perhaps 10° to a steep slope of perhaps 75° or so. Measure the angle of your ramp with a protractor (to the nearest degree is close enough) and enter the value in the box just under the title of Table I of the data sheet.

Now take a pencil and make a mark at the upper edge of the ramp exactly at the point where the upper edge crosses the face of the breadboard. Make a about 8 or 10 other marks along the length of the ramp between the first mark and the highest free point of the ramp. It doesn't matter exactly where you choose to place these marks: try to space them out to cover the whole free length of the ramp, but do not put them at carefully measured positions. Mentally number these marks from number 1 at the bottom of the ramp (where it crosses the face of the breadboard) consecutively upward along the ramp. Write these numbers in the first column of Table I. You are now ready to make your measurements. Check the ramp angle by protractor to make sure it hasn't moved.

Measure the height above the face of the breadboard of each mark on the ramp. Mark number 1 was made right at the point where the ramp crosses the breadboard, so its height is obviously zero. This is already recorded for you in the second column of Table I. Measure the heights of each of the other points in order. You can do this with a ruler, placing it so that the zero mark of the ruler is right at the ramp mark and allowing the ruler to hang downward from there. You must make sure that the ruler is perpendicular to the breadboard at the point where they cross. The square corner of a file card (or anything similar) will help assure that you get them perpendicular. The height is given by the point on the ruler where it crosses the face of the breadboard. Read it to the nearest 0.01 cm and enter the readings immediately in column 2 of the Table. Do this for each mark you made on the ramp.

Now unclamp the ramp and measure the "distance along the ramp" of each mark. To do this, notice that the "bottom of the ramp" was the point where it crossed the breadboard, mark number 1. Its distance from the bottom of course is zero, and this value is already entered for you in column three of the table. The "distance along the ramp" of any other mark is its distance from mark number 1. Measure each of these distances to 0.01 cm and record them in the third column. You are now finished with the measurements for the straight ramp. Check the angle once again with the protractor to make sure the ramp hasn't moved.

Repeat the entire procedure for the curved ramp. This time it is not necessary to measure the angle of the ramp, but repeat all other measurements exactly as for the straight ramp. Record the measurements in Table II. To measure "distances along the ramp" this time, you may find it convenient to lay a narrow strip of paper along the ramp so that the paper bends with the ramp, mark the paper where the ramp marks are, and then straighten the paper strip and measure the marks on it.

Finally, calculate "(height above ground) divided by (distance along ramp)" for each line of the two tables. You will notice that the first line of each table requires that you divide zero by zero. Leave this division unperformed for the time being.

Before completing this experiment (that is, the second work sheet), you will have a classroom discussion.

On the second work sheet, make a graph by plotting "distances along the ramp" in the horizontal direction and "height above ground" on the vertical axis. Marks for each cm of distance are already marked on the axes for you. Do this first for the data of Table I, drawing a tiny circle around each plotted point. Then, lightly in pencil, draw the best line you can through the plotted points. On the same axes, do the same for the crooked ramp, data of Table II.

In the first box at the bottom of work sheet #2, enter the value calculated from H = kD where D = 15.00 cm and k is the average ratio from Table I. In the second box, record the value of H when D = 15.00 cm as determined from the graph for the straight ramp, after it is redrawn.

Experiment 10
Data Sheet #1

Table I

Height and Distance along a Straight Ramp

Angle between ramp and "ground" = °

Mark #	Height above ground, cm	Distance along ramp, cm	Ratio $\equiv \frac{\text{Height}}{\text{Distance}}$
1	0.00	0.00	
2			
3			

Average Ratio

Table II

Height and Distance along a Crooked Ramp

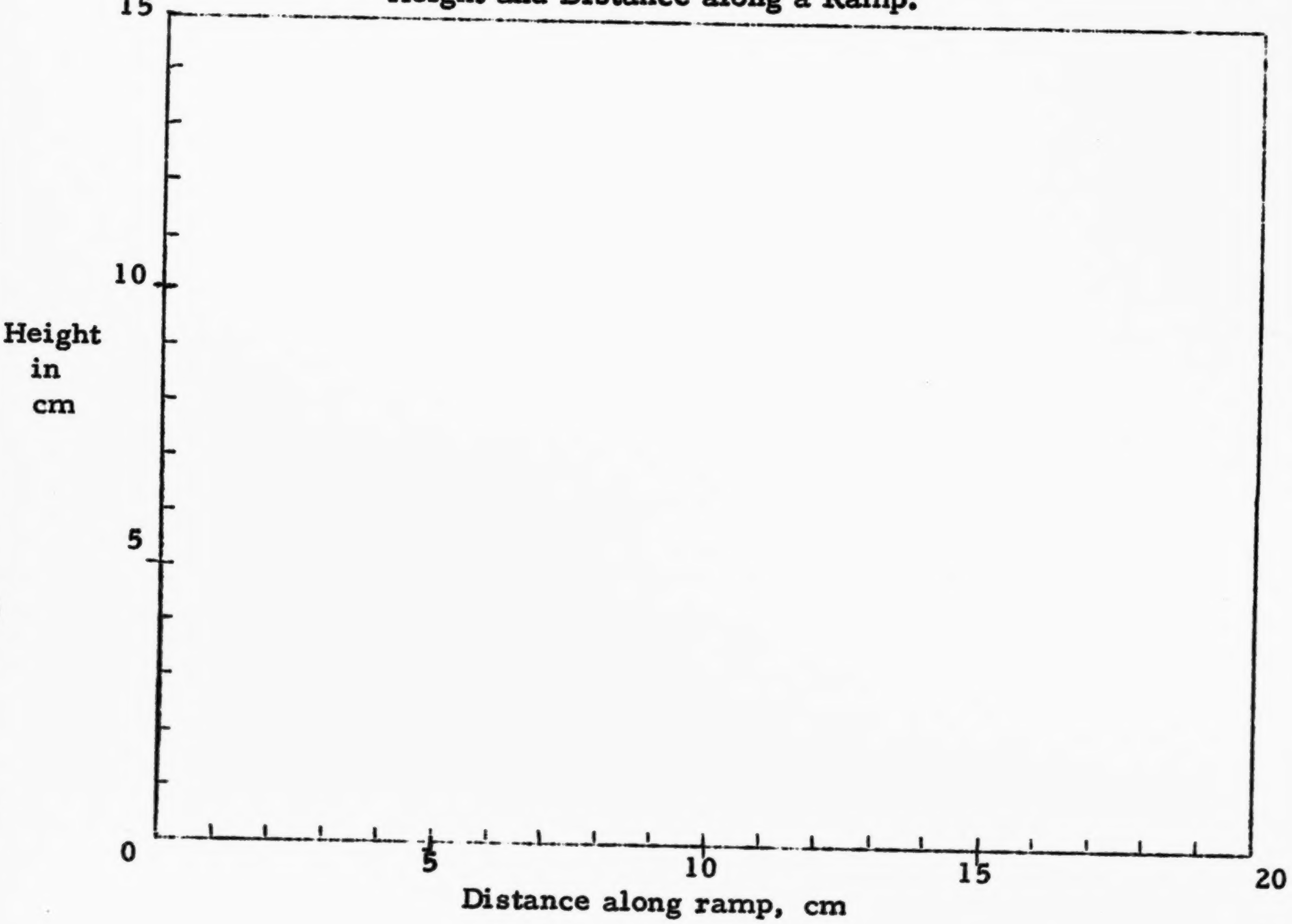
Mark #	Height above ground, cm	Distance along ramp, cm	Ratio $= \frac{\text{Height}}{\text{Distance}}$
1	0.00	0.00	
2			
3			

Average Ratio

Experiment 10
Work sheet #2

Figure 1

Height and Distance along a Ramp.



Plot the data from both tables of the preceding data sheet on the graph above. Label the two lines one "straight ramp" and one "crooked ramp."

For the straight ramp:

Calculated value of H for D = 15.00 cm - cm

Graphical value of H for D = 15.00 cm - cm

Experiment 11

Hooke's Law

We have several times referred to the functional relationship that we would expect to exist between the weight that is hung on a vertical spring and the amount by which the length of the spring increases. In this experiment you will examine the nature of this functional relationship. Be sure you understand exactly what it is that we are going to examine. If you hang up a spring and then hang a weight on its lower end, the length will increase. How much will it increase? Well, that depends upon how much weight you attach. The more weight you attach, the more the spring will extend. There is a functional relationship between "extension of spring" and "weight attached." The extension of the spring is a function of the weight that is added; that is, we will think of the added weight as the independent variable and the increase in length of the spring as the dependent variable, because the increase in length depends on the weight added. (We mean by "increase in length", not the actual length of the spring, but how much the actual length exceeds the length when no weight is attached.)

Procedure: You will be supplied with a dowel-post, dowel-pin, breadboard, 20 ball bearings, a piece of scotch tape, a ruler, and a spring. Set up the breadboard, post, pin, and ruler like the model your teacher has already set up. Use small pieces of scotch tape to attach the ruler (zero end at the top) to the vertical post, but be sure the ruler is securely held in place. Hang one end-hook of the spring over the dowel-pin, attach a two-inch length of scotch tape to the lower hook so that the open sticky surface hangs downward. Be sure the tape is securely attached. Record the number of your spring at the top left of Table I, where it says "Spring No."

Now sight horizontally across the top of the upper hook to the ruler, and take a reading of the position of the upper hook as shown by the ruler. Make all readings of the ruler to the nearest 0.01 cm. Record this just below "Spring No." in Table I where it says "Pos'n of upper hook." Then sight horizontally across the bottom of the lower hook to the ruler and read its position as shown by the ruler. Record this reading on the first line of the table where the entries "O" balls and "O. OO" grams have already been made. Now carefully attach a ball bearing to the scotch tape, let the spring come to rest, and read again the position of the bottomof the lower hook. Record this reading on the second line of the table opposite "1" ball. Now take about ten more readings by attaching successively more balls to the tape and each time reading the position of the bottom of the lower hook, the last reading with 20 balls. Each time, record in the first column the total number of balls sticking to the tape, and in the third column the position of the bottom of the lower hook.

In making readings on the ruler of the position of the hook, it is very important that you sight horizontally across the hook to the ruler. Do you see why?

After you have completed your readings on this spring, turn it in to your teacher and obtain a second, stiffer spring. Repeat the whole experiment with the second spring, recording the readings in Table II.

Now fill in the second column of the table, the weights successively hung on the spring. The ball bearings are all alike in weight. Your teacher will tell you this weight, which you should record in the upper left box of each table. You now can calculate the load hung on the spring by multiplying the weight of one ball by the number of balls attached. Record these loads in the third column of the table.

Next calculate the length of the spring for each load you applied. You have the ruler-position of the top of the upper hook and of the bottom of the lower hook. How would you calculate the length of the spring from these data? Notice that the position of the upper hook never changes, so it need be read only once. Record your calculated values of "Length of Spring" in the fourth column.

Next calculate the spring extension for each load. To do this, remember what is meant by "extension." The extension of the spring is the amount by which the length of the spring under load exceeds the length under no load. The first entry in column four, is the unloaded length. Other entries in column four are the loaded lengths. Do you see how to calculate the extension now for each di different load? Do so. The extension for zero load of course is zero, and this value is already entered for you.

Next, calculate the ratio, "Extension/Load", for each line of the table.

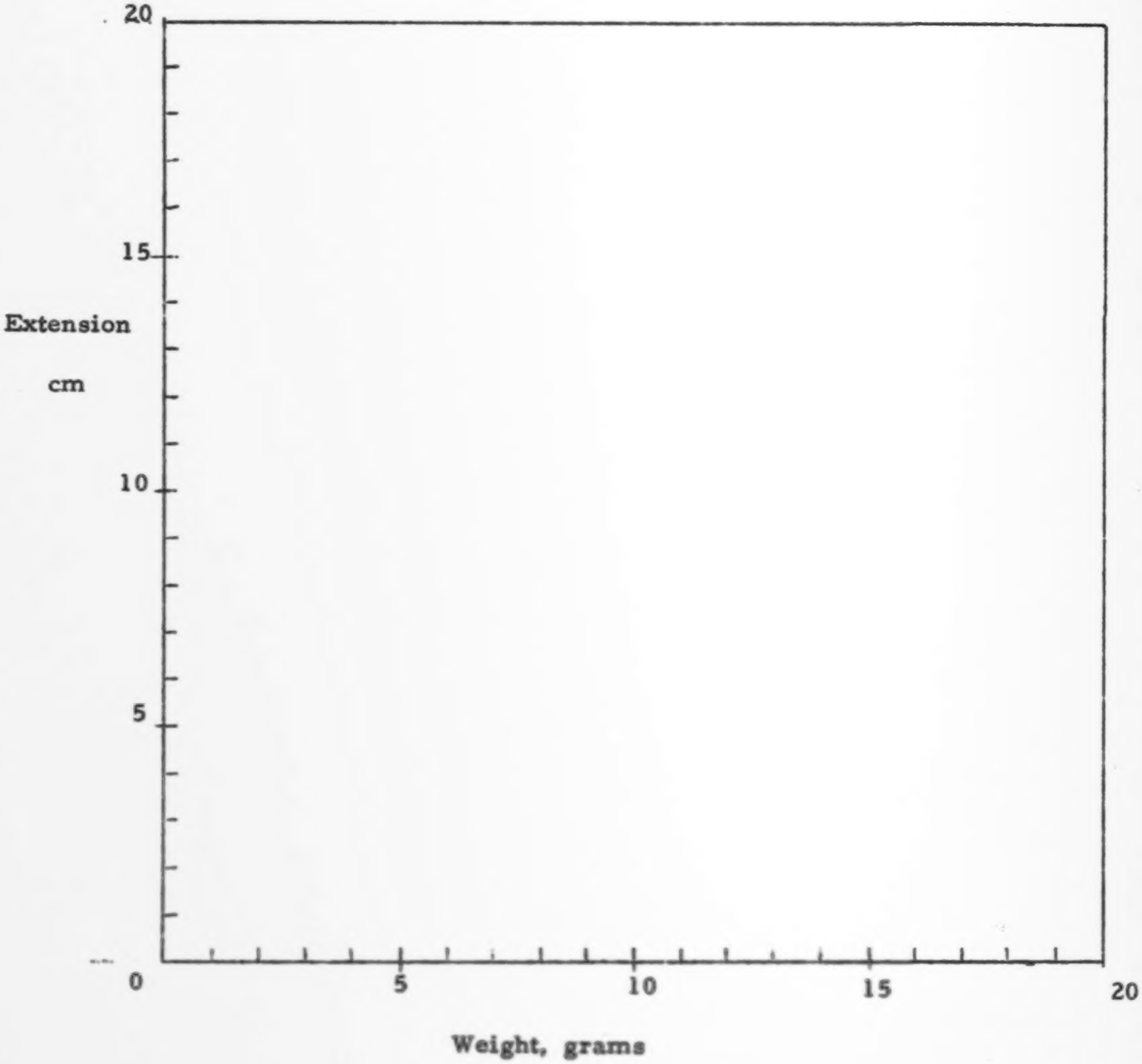
Make all these calculations for both tables.

Finally make a graph of extension (vertically) against load (horizontally) for both springs. This graph is on the second worksheet.

Experiment 11
Data Sheet

Weight of one ball = grams		Spring No.			
		Position of upper hook cm			Extension / Load g/cm
No. of Balls	Weight g	Position of lower hook, cm	Length of Spring, cm	Extension cm	
0	0.00				

Experiment 11
Work Sheet #2



Spring Extension vs. Weight Load

Experiment 12

Weight as a Function of Length for Uniform Sticks

In this experiment, we will investigate the nature of the function connecting the weight of an aluminum rod of fixed diameter but varying length with the weight of the rod. The two variables between which we are seeking a functional relationship are the weight of the rod and its length.

Procedure: Set up the breadboard with dowel post, pin, and ruler exactly as in Experiment 11. Use the same spring that you used before, and record its spring constant in the space provided near the bottom of the data sheet. You will be furnished with a short piece of thread and a set of aluminum rods all of the same diameter (0. 635 cm). Fashion a sling out of the thread so that the rods may be hung one at a time from the end of the spring.

Make a reading of the bottom of the lower hook (nearest 0. 01 cm) with no load hanging on the spring, and record this reading in the space provided near the bottom of the data sheet. Then hang one of the aluminum rods on the spring, allow it to come to rest, and read again the position of the hook. Make all readings to the nearest 0. 01 cm. Record this reading in the third column of Table I of the data sheet. Then measure the length of the rod, also to the nearest 0. 01 cm, and record this reading in the second column of the table. Record the number of the rod in the first column. Repeat these measurements for at least 8 rods of the same diameter.

You now have the position of the bottom of the spring when it is unextended (in the box at the bottom left of the data sheet) and the position when it is extended (third column). How can you calculate the extensions? Do so, and enter them in the fourth column of the table. Using the spring constant you can now calculate the weights that must have caused these extensions. Calculate these weights and enter them in the fifth column. Then calculate the ratio of weight divided by length and write the calculated ratios in the last column.

When you have completed all measurements on the rods 0. 635 cm in diameter, obtain another set measuring 0. 318 cm in diameter and repeat the whole experiment with them, recording your data in Table II.

Experiment 12
Data Sheet #1

Table I

Lengths and Weights of Some Aluminum Rods
(0.635 cm diameter)

Rod No.	Length cm	Bottom of Spring, cm	Extension cm	Weight grams	$\frac{\text{Weight}}{\text{Length}}$ g/cm

Table II

(0.318 cm diameter)

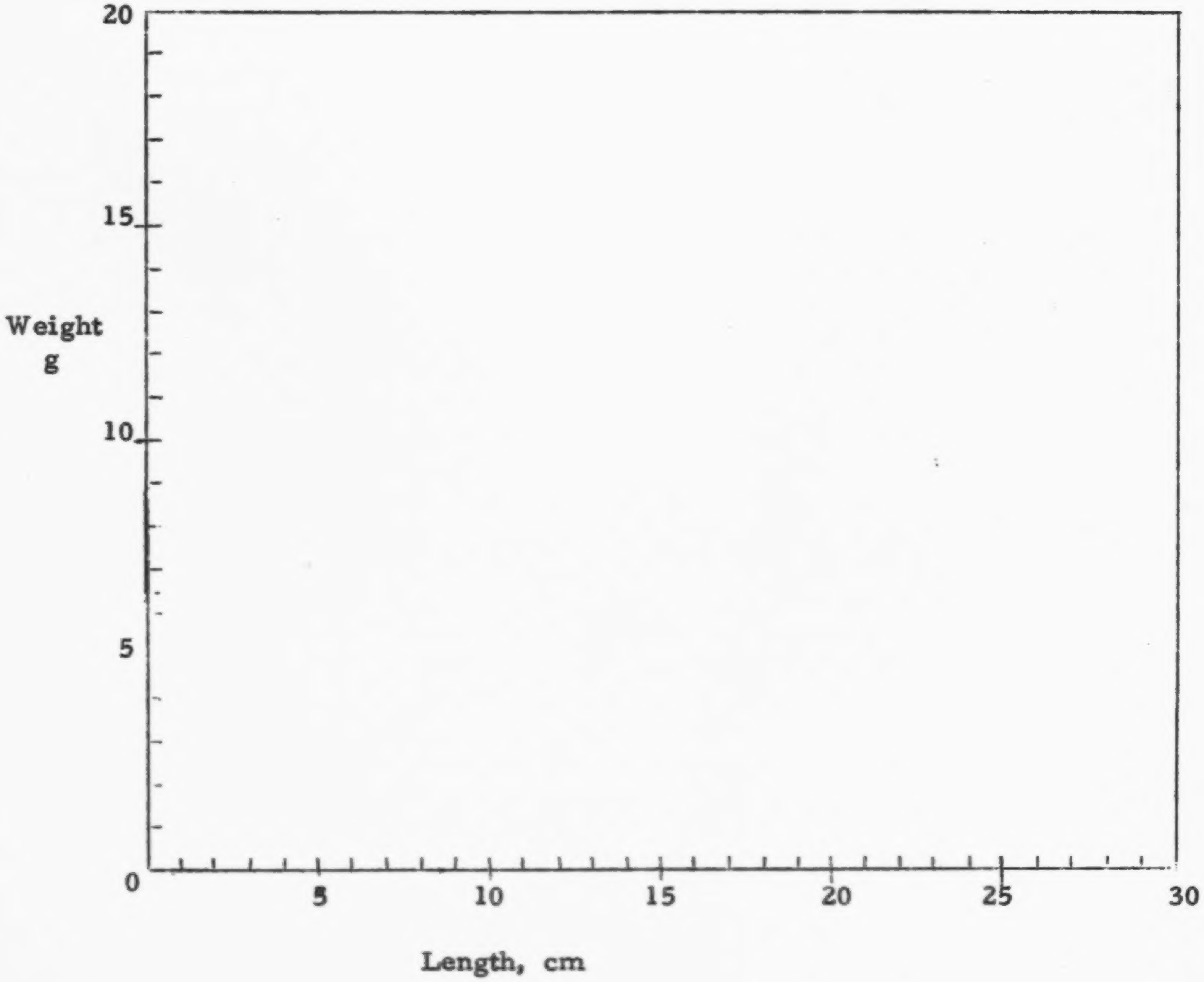
Rod No.	Length cm	Bottom of Spring, cm	Extension cm	Weight grams	$\frac{\text{Weight}}{\text{Length}}$ g/cm

Unextended Position of Spring End ________ cm

Spring Constant ________ cm/g

Experiment 12
Work Sheet #2

Weight vs. Length for Aluminum Rods of Different Diameters



Experiment 13

Weight as a Function of Diameter for Cylinders of Fixed Length

The purpose of this experiment is to investigate the functional relationship between the weight of an aluminum cylinder of fixed length and its diameter. You will determine the weight of each one of a set of cylinders all having a length of 2.54 cm but of different diameters, and then seek a functional relationship between weight and diameter.

Procedure: Set up the apparatus exactly as in Experiment 12. Be sure you use a spring whose constant is known, recording the value of the constant in the appropriate place on the data sheet. Prepare a sling of thread that will allow you to hang each cylinder individually. Read the position of the bottom of the hook when no load hangs on the spring, and record the reading on the data sheet.

You will be supplied with a set of aluminum cylinders, all 2.54 cm long, but of varying diameters. Measure the diameter of each cylinder (use the caliper method) to 0.01 cm and also the point to which it extends the spring when hung upon it. Record both data in the proper columns of Table I. Do this for at least 8 cylinders, being sure that they are all different in diameter.

After you have completed your measurements, compute the ratio of weight/diameter for each line of Table I, entering the ratios in the second-last column of the table. Leave the last two columns blank for the time being.

Now make a graph of weight versus diameter at the bottom of the data sheet, plotting diameter horizontally and weight vertically.

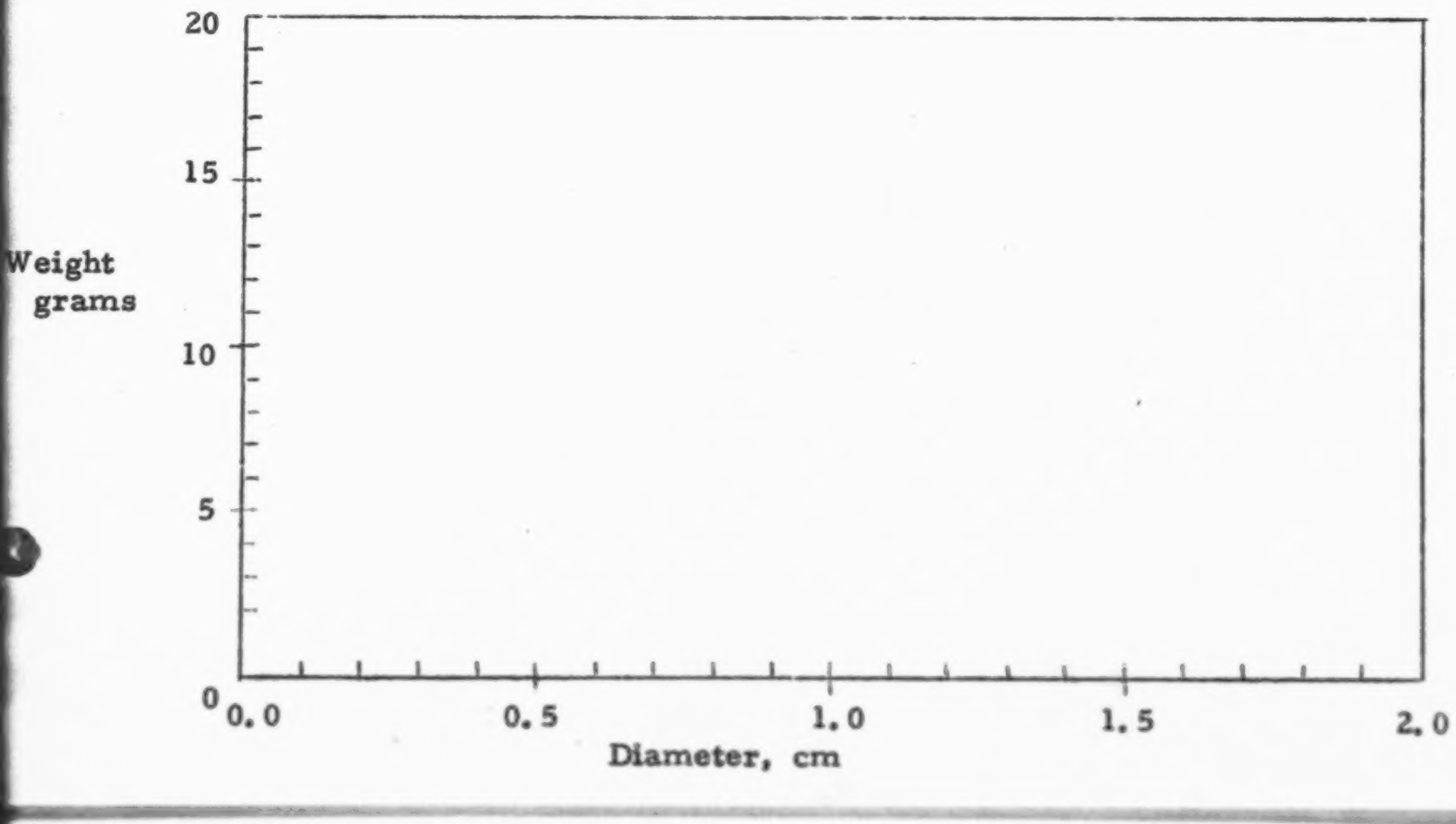
Experiment 13
Data Sheet

Table I
Weights and Diameters for Some Aluminum Cylinders
(2. 54 cm long)

Cyl No.	Dia cm	Bottom of Spring, cm	Extension cm	Weight. grams	$\frac{\text{Weight}}{\text{Diam.}}$, g/cm		

Unextended Position
of Spring End cm Spring Constant cm/g

Weight versus Diameter for Aluminum
Cylinders of Fixed Length



Experiment 14

Weight as a Function of Diameter for Aluminum Spheres

In this experiment you will try to find a functional relationship between the weight of an aluminum sphere and its diameter. Be sure you see exactly what the two variables are, between which we are seeking a functional relation.

Procedure: The set up is exactly like that in Experiment 13 except that you will probably find it more convenient to hang the spheres from the spring by using scotch tape instead of a sling made from thread. Enter on the data sheet the spring constant for yourcalibrated spring and the initial unloaded position of the bottom of the spring.

You will be provided with a set of 8 aluminum spheres of different diameters. Take one of these spheres and weigh it by hanging it from the bottom of your spring and reading the position of the bottom of the hook. Enter this reading in the second column of Table I on the data sheet. Then use the caliper method to determine the diameter of the ball, recording this measurement to the nearest 0.01 cm in column one of the table. Repeat the measurement for all 8 of the spheres provided. Calculate the weight of each ball by computing first the extension (column 3) and then the weight (column 4) in the usual way. Next compute the ratio of weight/diameter and enter these values in column 5. Plot weight vs. diameter on the graph at the bottomof the data sheet.

Now, before doing anything more with the data from this experiment, we will have some classroom discussion.

Experiment 14
Data Sheet

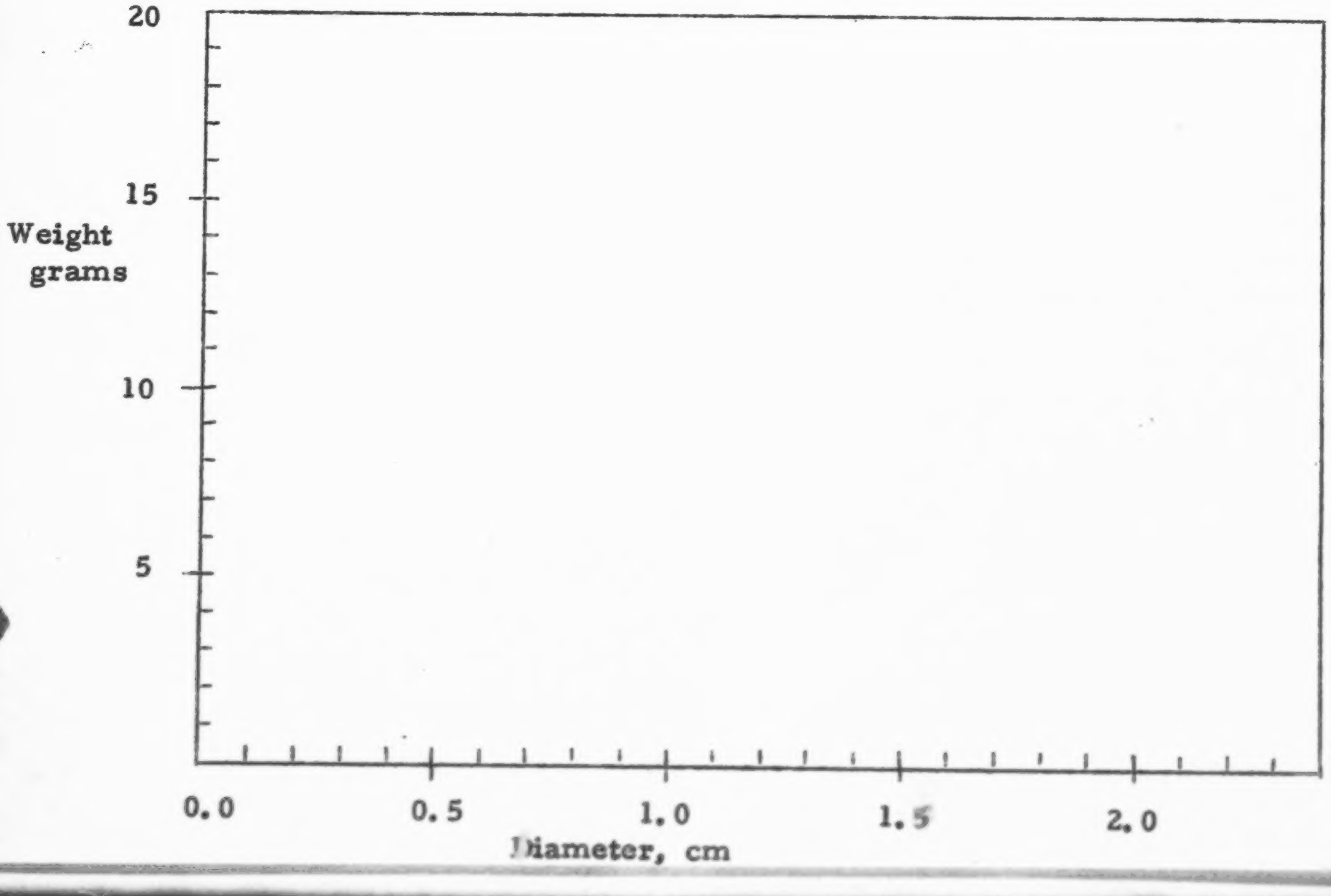
Table I

Weight vs. Diameter for Aluminum Spheres

Diam. cm	Sp. Pos. cm	Extension cm	Weight g	Wt/Ext g/cm				

Unextended Position of Spring End cm

Spring constant cm/g



Experiment 15

Density of Aluminum

You know from your own experience that larger things are generally heavier than smaller things. Not always, of course. A small piece of lead may be heavier than quite a large balloon, for instance. But if the two things under comparison are made of the same material, the ststement is generally true that the larger the object, the more it weighs. You would predict that the weight of a piece of iron is an increasing function of its size.

But what do we mean by "size"? Is a piece of iron wire 10 cm long and a hair's thinkness in diameter bigger than a ball of iron 9 cm in diameter? Is the lump of brass in a solid ball 3 inches in diameter smaller than that in a thin hollow brass ball 4 inches in diameter? Is a matchbox measuring 2 cm by 5 cm by 3 cm bigger or smaller than a cube measuring 3 cm each way? Is a size 8 shoe bigger than a size 7 hat? You notice that we use the word "size" rather imprecisely in our ordinary speech. But we cannot afford such imprecision when we begin dealing with the numerical aspects of quantity.

We must use words so that none of the questions in the preceding paragraph is arguable. We will do this by avoiding the word "size" altogether and use the word "volume" in its place. If you consider each of the above questions as dealing with volumes, each of them has a very definite answer (though of course you may not offhand know the answer).

The purpose of this experiment is to investigate the nature of the function involved when you say "The weight of a lump of aluminum is an increasing function of the vplume of the lump."

Procedure: The set up for this experiment is exactly like that for Experiment 14. Again, you will probably find it more convenient to use scotch tape than a sling of thread. Record the spring constant and the initial reading of the unextended spring in their proper places.

You will be furnished eight small blocks of aluminum of various shapes and sizes. Each will be either a rectangular block or a circular cylinder. You are to determine the weight and volume of each piece.

To find the volume of a rectangular block, measure the length, width, and height, each to 0.01 cm. Record the number of the block in the first column of Table I of the data sheet and in the second column record the shape and dimensions like this:

Rectangular Block
1.76 x 3.41 x 1.32 cm

For the cylindrical blocks, you must measure the height of the cylinder and the diameter. Measure the latter by the caliper method, length and diameter both to 0.01 cm. Record the numbers of the block in the first column and a description of its shape and size in the second column like this:

Cylinder
Diameter = 2.41 cm
Height = 3.66 cm

After you have measured and recorded the dimensions of you first block, stick it to the scotch tape hanging from your spring, allow the spring to come to rest, and measure the position of the bottom of the lower hook. Record this measurement in the fourth column of the data table.

Repeat these measurements for eight different blocks, about half rectangular and half cylindrical.

Now compute the volumes of your blocks. For a rectangular block, the volume is the product of length times width times height. For a cylindrical block, the volume is 1/4 x π x diameter x diameter x height. In what units are these volumes and how many significant figures are you entitled to? Record the volumes in the third column.

Compute the weight of each block from the spring extension and spring constant in the usual way, and enter the computed weights in the sixth column of the table. Compute the ratio of weight/volume and record the ratios in the last column. When you are finished make a graph of weight versus volume on the second work sheet.

Experiment 15
Data Sheet #1

Table I

Weights and Sizes of Some Aluminum Blocks

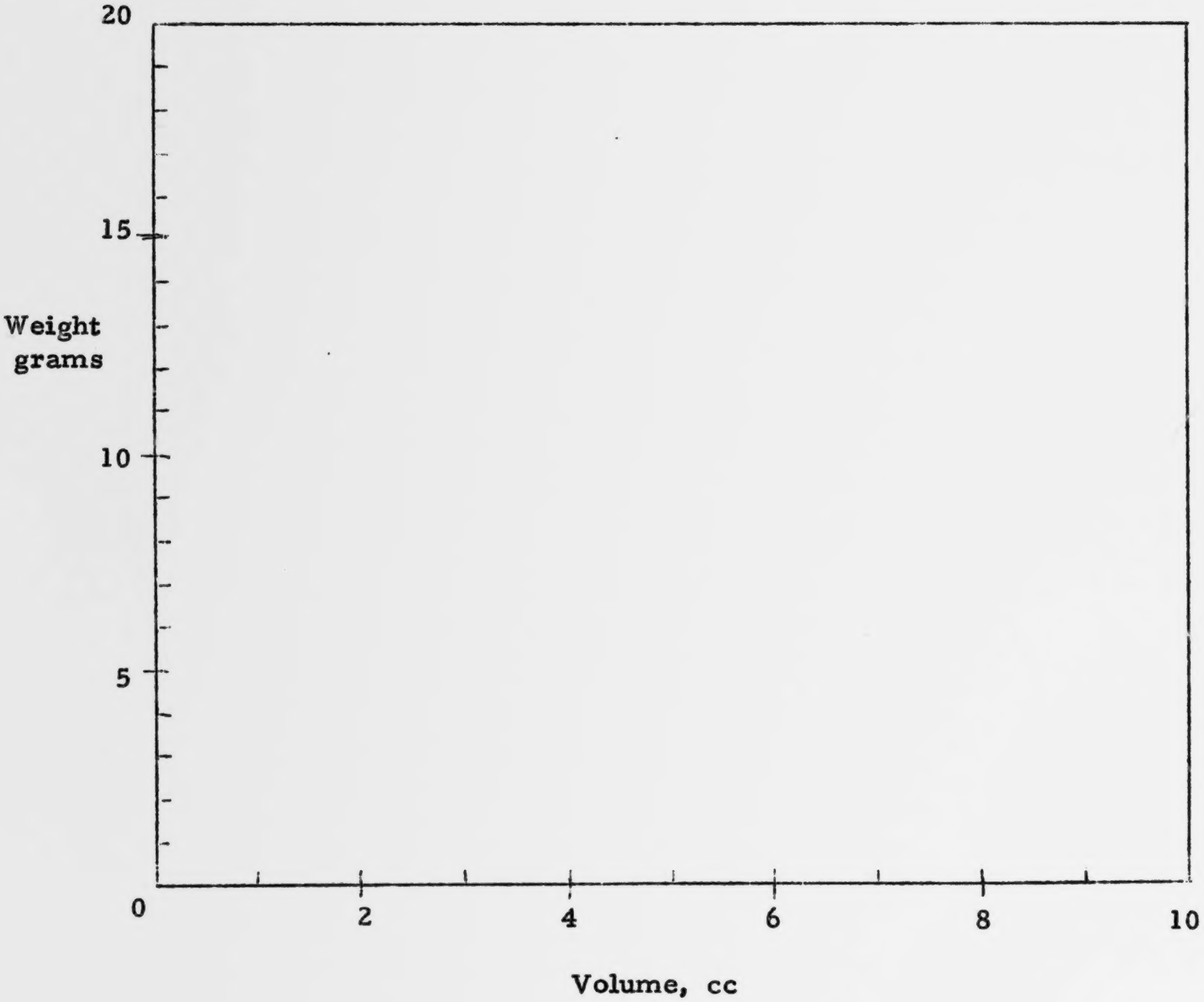
Blk. No.	Shape and Dimensions	Volume cm^3 or cc	Sp. Pos. cm	Extens. cm	Weight grams	Ratio g/cc

Average Density of Aluminum = ______ g/cc

Unextended Position of Spring End:	cm
Spring constant =	cm/g

Experiment 15
Graph Sheet #2

Weight vs. Volume for Aluminum



Experiment 16

The Densities of Various Solids

In this experiment, you will obtain data from which you will be able to compute the density of each of several materials other than aluminum. The purpose of Experiment 15 was to establish that weight is proportional to volume, and to do so it was necessary to measure a relatively large number of blocks to establish that the ratio of weight/volume is constant. Having done this once (for aluminum) there is little point in making so many measurements again just to find one density. Hence, to save time and effort in the present experiment, you will make measurements on only two blocks of each material.

Procedure: The setup is identical with that of Experiment 15. Make the measurements in the same way and record them in the same way on the data sheet.

With one exception! Up to this point you have been using the spring constant in cm/gm and dividing the extension (cm) by the constant (cm/gm) to get the weight (gm). (Can you still show that dividing cm by cm/gm gives gm?) You know that dividing by a number is the same as multiplying by its reciprocal. Most people find it easier to multiply than divide. Has it occurred to you that you can make the work a little easier by using the spring constant in gm/cm, and then multiply the extension by the new constant to get the weight? The new spring constant in gm/cm is, of course, the reciprocal of the old one in cm/gm. You will have to divide the old constant into 1 (How many significant figures in this 1?), but after that one division, all the rest will be multiplying.

You will be furnished with two blocks each of wood, plastic, brass, steel, and lead. Be very careful of the blocks (especially the lead ones) so that the edges remain sharp and easily measured -- mashed edges cannot be measured accurately. The name of each material is already entered in the first column; be sure you put the data for each measurement on a line opposite the appropriate name. Calculate the densities for each line and put in the last column the average of the two measurements you made for each material.

Make graphs of weight vs. volume for all five materials. Put all five curves on the same graph on the second work sheet. You have three points to outline the curve for each material: the two measured points plus the origin. Is this enough, or more than enough, to show where the straight line for that material lies?

Experiment 16
Data Sheet #1

Table I.

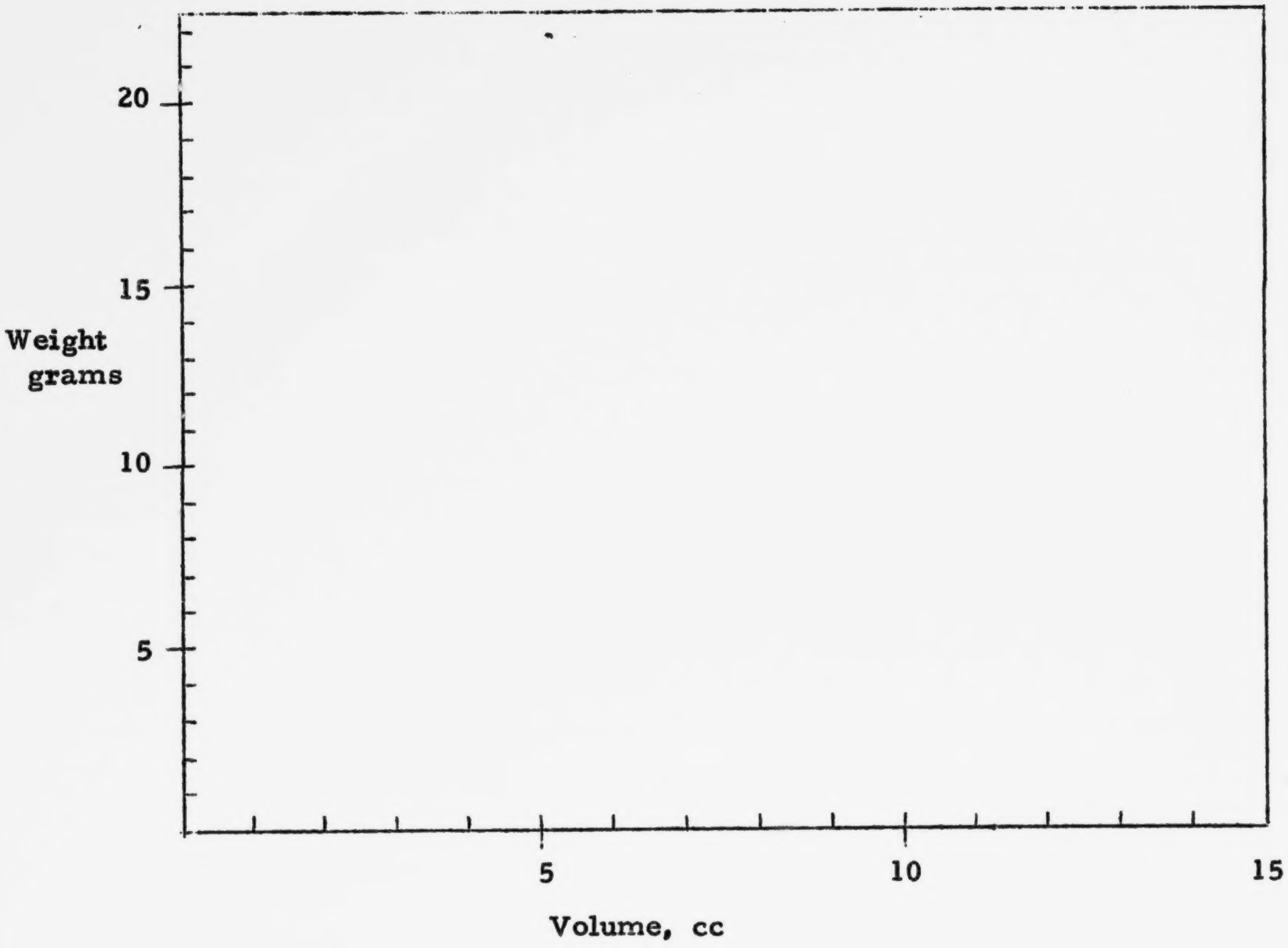
Densities of Various Solids

Material	Shape and Dimensions	Volume cc	Sp. Pos. cm	Exten. cm	Weight g	Density g/cc	Average Dens.g/cc
Wood							Wood
Plastic							Plastic
Brass							Brass
Steele							Steele
Lead							Lead

Unextended position of Spring End	cm
Spring Constant	g/cm

Experiment 16
Graph Sheet #2

Weight vs. Volume for Various Solids



Plot the points and draw graphs, one for each of the materials whose densities you measured. Label each curve with the name of the material it represents.

Experiment 17

Identification of Unknown Solids by Density

This experiment tries to show you how a knowledge of the properties of materials makes it possible for you to identify an unknown material. The idea is not new to you: you recognize glass from iron, water from salt, waxed paper from aluminum foil, and hamburger from pickles by an automatic recognition of the differences in their properties. If you were given some granulated sugar and some granulated iron, you could tell the difference right away by the color. Suppose you had sugar and salt; color doesn't help you decide, but taste will. Suppose you had granulated sand and sugar, and you were afraid to decide which was which by tasting. How could you safely decide? Suppose you had sand and granulated marble, neither of which will dissolve in water? This is less easy, but it happens that marble will dissolve in vinegar but sand will not. One could go on like this testing property after property until some property was found where the two disagreed. This experiment will use density only.

You will be furnished with 7 blocks of material; one of each of the materials used in Experiment 16 and one that is none of these. They are all painted black, however, so that it may not be easy to tell them apart at sight. Determine the density of each block by the same procedure you used in Experiment 16. One determination for each is enough. Before you do this, you should use a little educated guesswork to try to decide. Which ones feel cold to the touch? Can you tell anything fromthe surface appearance? What about the heft? Do NOT attempt to scratch any of them: not only is this cheating, but it would also damage some of the blocks!

Make up you own data sheet on the next page. Include in it the number of the block and the material you judge it to be after measuring its density. Also be sure to record the no-load spring extension and the spring constant in gm/cm. Your teacher will identify the materials for you after you have made your own decision.

Experiment 17
Data Sheet

Experiment 18

The Densities of Some Liquids

To calculate the density of a material, you need have simply the weight and the volum of a sample of the material. When the sample is in the form of a simple geometric solid, it is a simple matter to measure its dimensions, and compute its volume, then weigh it, then compute the ratio, weight/volume. With liquids, however, it is less simple to make the necessary measurements. For one thing, you have to have the liquid in some kind of container when you weigh it, and then you are bothered by the weight of the container. For another thing, it is difficult to measure the dimensions of a piece of liquid, because the sample won't hold still for you while you measure it. In this experiment, nevertheless, you will determine the densities of several liquids.

Procedure: Set up a calibrated spring and post again as a weighing instrument. You will be furnished a plastic vial with cap. Tie a short piece of thread in the form of a sling (similar to the handle of a bucket) securely to the vial so that it may be hung thereby from the bottom of the spring. Be sure the thread is tied low enough below the rim so the cap can be fitted to the vial, but not so low that the vial may tip over when it hangs by the sling. Hang the vial on the lower hook of the spring, cap it, let the spring come to rest, and carefully measure the position of the bottom of the spring. to the nearest 0.01 cm. Enter this reading on the data sheet as "Unextended position of spring." Also enter the spring constant in g/cm in the proper place.

We will measure volumes of liquid by using an instrument called a pipette. A pipette is simply a hollow cylinder (that is, a straight tube) whose inside diameter is accurately uniform. When the tube holds a liquid, then, the liquid itself is in the form of cylinder, bounded on its curved outside by the inside wall of the tube, at one end by the bottom of the tube, and at the upper end by the free surface of the liquid. (Do you picture this cylinder in your mind?) If you knew the inside diameter of the tube and the length of the column of liquid, you could then calculate the volume of the liquid. If this volume came to, say, 9.76 cc, you could then put a mark on the tube saying "whenever the pipette is filled to this level, the volume of liquid it contains is 9.76 cc." Actually you can buy pipettes that are already calibrated this way, graduated as a ruler is graduated, showing not lengths, but volumes of liquid contained. A pipette is filled to any desired level in the same way that a drinking straw is filled, and is kept from emptying itself by pressing a finger against its upper end. Your teacher may prefer, for sanitary reasons, to do the pipetting for you.

Now pipette into the vial hanging on your spring an accurately measured volume of water somewhere in the neighborhood of 15 cc. Many beginners are tempted, in this kind of operation, to go to great trouble to adjust the contents of the pipette to exactly 15.00 cc. This is a foolish waste of time. Any volume of water somewhere near 15 cc (say anywhere from 13 to 17 cc) is as good as any other. The point is that you may use any volume of water you happen to get in the pipette, but whatever volume you use must be measured accurately. Read the volume of water to the nearest 0.01 cc, and record it on the first line of your data sheet, column one. [Your teacher may prefer to have you bring your vial to a central place to receive a sample of water. If so, carefully detach the thread bail from the spring, leaving the thread on the vial, have the water placed in it, and then hang it again from the spring.]

To prevent evaporation of the liquid once its volume has been measured, keep the cap on the vial at all times except when you are actually adding or removing liquid. Allow the spring to come to rest and read the position of the bottom of the hook once more. Record this reading in column two of the data sheet. Empty the vial and dry it, obtain another sample of water, and repeat the experiment. Enter volume and spring position again in the table, the new data on the second line.

Now repeat this experiment using alcohol instead of water. After you have finished with your first sample of alcohol, however, do not throw it away as you did the water. Pour it into the bottle for waste alcohol as designated by your teacher. Make two separate measurements of volume and spring position on two different samples of alcohol, recording your observations in the first two columns, last two lines of Table I. Repeat with benzene and carbon tetrachloride. For carbon tetrachloride, use about 10 cc samples. (Carbon tetrachloride slowly attacks the plastic of the pipettes and the vials. Do not let them stand in contact more than a few minutes at a time. Alcohol, benzene, and water are without effect.)

For all eight lines of the table, calculate columns 3 and 4 in the usual way. Then compute the density (column 5) by dividing weight by volume. Record the computed densities (three decimal places for each) in column 5. The two measured densities for water should agree closely, and also the two for each of the other liquids. Average the two values for water and enter the average in the last column. Do the same for each of the other liquids.

NOTE:

> There is no special danger involved in the use of these liquids, but they are all poisonous (except water) if you drink them. Getting them on the hands is not harmful, but you should avoid doing so anyway. Alcohol and benzene are inflammable; there should be no flames in the room where this experiment is done. It is harmless to smell the liquids, but avoid deliberate breathing of the vapors, especially benzene and carbon tetrachloride.

Experiment 18
Data Sheet

Table I

Densities of Several Liquids

Substance	Volume cc	Spring Position	Extension cm	Weight g	Density g/cc	Average Density
Water						Water g/cc
Alcohol						Alcohol g/cc
Benzene						Benzene g/cc
Carbon-Tetrachloride						Carbon Tet. g/cc

No-load Spring Extension	cm
Spring Constant	g/cm

Experiment 19

Densities of Alcohol-Water Mixtures

The density of water is greater than that of alcohol, and the two liquids readily mix together. If you make a mixture of the two, the mixture will be neither alcohol nor water, and probably would then have the density of neither. If you take a very large amount of alcohol and add to it only a drop of water, how would you expect the density of the mixture to compare to the density of pure alcohol? If you add a drop of alcohol to a very large amount of water, how would you expect the density of the mixture to compare to that of pure water? How are your answers illustrative of the principle of continuity?

If you start with a glass of water and add alcohol to it drop by drop, what does the principle of continuity say about how the density of the mixture changes from one drop to the next. If you keep on adding alcohol until you have a tank-car full of mixture, you would end up with practically pure alcohol wouldn't you.? You would therefore have made a series of mixtures ranging all the way from pure water (density = 1.00 g/cc) to pure alcohol (density = 0.79 g/cc). What would the principle of continuity say about the densities of these mixtures?

The purpose of this experiment is to investigate how the density of an alcohol-water mixture changes with composition. We are looking for a functional relationship between density (the dependent variable) and composition (the independent variable). This raises and important question: density is of course a numerical quantity, but is composition? Put it this way: How much of this sample of alcohol is alcohol? All of it, you say; 100 per cent of it; not half of it, not three-fourths of it, not 99/100 of it, but 1 of it. In considering mixtures of alcohol and water, we would say that pure alcohol had a "fraction" of alcohol equal to 1. Pure water would have a fraction of alcohol equal to 0. If I mixed 10 grams of alcohol with 10 grams of water, the fraction of alcohol would be 0.50. If I mixed 7 grams of alcohol with 3 grams of water, I would have 10 grams of mixture of which 7 grams or 7/10 is alcohol. The fraction of alcohol is 0.70. If I mix W grams of water and A grams of alcohol, how many grams total? What fraction is alcohol?

We will use "fraction of alcohol" as a quantity representing the composition, and you should understand why this fraction is given by A/(A + W).

Procedure: This will be a cooperative experiment. The class will be divided into small teams so that there are nine teams. Each team will determine the density of one mixture, so that the whole class will determine the densities of nine different mixtures. Each team should set up a weighing apparatus as with the preceding experiments, using a calibtrated spring. The team will be furnished two vials, one of which should be slung from the string by a thread bail as before. Read the position of the spring with the capped vial hanging in place, and record the reading as "No-load position of spring" in Table II. Also record the spring constant in g/cc there.

Leave the vial hanging on the spring, and take the other vial and cap to your teacher and have a sample of water pipetted into it. Record the volume of water, measured to 0.01 cc, in the first column of the table. Then have a sample of alcohol pipetted into the same vial (right in with the water already there) and record this volume in the third column of the table. Cap the vial right away. Each team will get a different combination of water and alcohol as follows:

Team #	Approximate Volume of	
	Alcohol	Water
1	18 cc	2 cc
2	16	4
3	14	6
4	12	8
5	10	10
6	8	12
7	6	14
8	4	16
9	2	18

As before, the volumes measured should be approximately those in the table but need not be exactly these. Whatever they are, however, they must be measured precisely to the nearest 0.01 cc. Notice that volumes of more than 10 cc will require two fillings, and therefore two readings, of the pipette. Add the two readings together for the total.

Now swirl the capped vial around for a minute or two, gently but thoroughly, to mix the alcohol and water completely. DO NOT SHAKE the vial, for you must avoid getting liquid under the cap. Then detach the threaded vial, with thread, from the spring and take both vials to your teacher. Your teacher will use the pipette now to remove about 17 or 18 cc of the mixture from the first vial and transfer it to the threaded one. This of course will have to be done in two fillings and readings of the pipette. Read each filling to 0.01 cc and add the two together to get the volume of mixture now in the threaded vial. Record this volume in the eighth column of the table under "cc of Mixture." Cap the threaded vial.

Discard the remaining liquid in the plain vial and hang the capped threaded vial back on the spring. Allow the spring to come to rest and read the position of the bottom of the hook. Record this reading in the third line of Table II. Calculate the extension of the spring from the two readings recorded in this table and enter it on the fourth line. Then, using the spring constant already recorded in the table, calculate the weight of the mixture. Enter this value in the last line of Table II and also in the second last column of Table I. Notice that your own work will fill only the first line of Table I.

You now know the volume of mixture (column 8) and its weight (column 9). Compute the density of your mixture and record its value in the last column, first line. You are now finished with your part of the experiment. When everyone has finished, your teacher will assemble the data of all teams. Copy the data from other teams on succeeding lines of the table, making sure you don't repeat your own.

Finally, make a graph on which you plot density (vertically) versus "fraction of alcohol" (horizontally). The two columns of the table headed by arrows (↓) show which to plot. The data of Table II give you nine points for the graph. You can get two more (zero and one fractions of alcohol) from the results of Experiment 18. Be sure to include these two points on your graph.

Experiment 19
Data Sheet #1

Table I

Densities of Water-Alcohol Mixtures

↓ ↓

Amt. Water		Amt. Alcohol		g Total	Fraction of		cc of Mixture	Wt. g	Density g/cc
cc	g	cc	g		Alc	Wat			

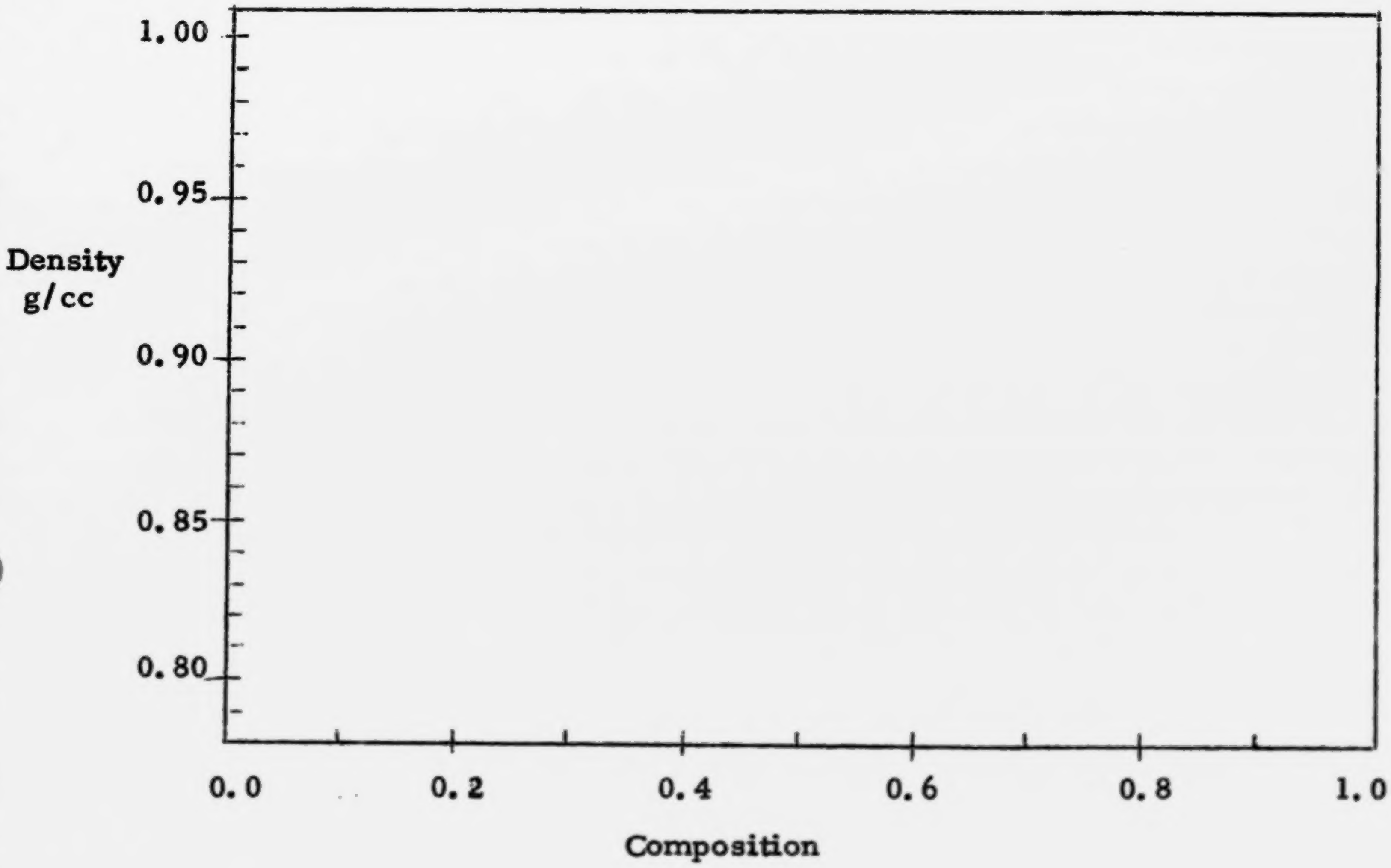
Table II

Spring Data

Spring Constant	g/cm
No load position of spring	cm
Loaded position of spring	cm
Extension of spring	cm
Weight of sample	g

Experiment 19
Graph Sheet #2

Density vs. Composition for
Mixtures of Alcohol and Water



An Example

Weight of 10 cc of water ____________ g

Weight of 10 cc of alcohol ____________ g

Total Weight ____________ g

Composition .____________

Density of this compositon from graph ____________ g/cc

Volume corresponding to total weight ____________ cc

Experiment 20

Density vs. Concentration for Sugar Solutions

You probably have a feeling that the density of a solution of sugar would increase as you dissolved more and more sugar in it. The purpose of this experiment is to look for a functional dependence between the density of a sugar solution and the concentration of the solution. We will think of the concentration as the independent variable and density as the dependent. Also, we will express the concentration of the sugar solutions in "grams of sugar per cc of solution". Be sure you see both that this is a concentration and also what is meant by "grams of sugar per cc of solution."

Procedure: This experiment also will be a team effort. Each person in the class will make up one sugar solution and determine its concentration and its density. Everyone will make a solution of a different concentration, so that the whole class together will have a complete set of data from a solution of very lowconcentration to one of very high concentration.

Set up a weighing apparatus with the spring arrangement you have used so often now, with a vial hanging from the spring with a thread. Record the spring constant on the first line of Table I. Take a reading of the bottom of the spring as usual (with the capped vial attached), and record it on the first line of the left-hand portion of Table I.

Detach the vial (with thread bail) and take it to your teacher to obtain a sample of sugar. The sugar can be conveniently measured with a "spoon" made by cutting long notches near the end of a Popsicle stick in such a way as to produce a spade-end about 1/2" long. Such a measure will hold about 1/3 gram of granulated sugar when used as a spoon. The exact amount it holds is unimportant because you are going to weight the sample accurately anyway. The first person to get his sample will receive 30 measures, the next 29, and so on down to 1. If the number of persons in the class is not exactly thirty, it is quite all right if some numbers are duplicated or if some numbers are missing. The samples should, however, span the range from 1 to about 30.

When you have received your sugar, cap the vial and take it immediately to your weighing machine and weigh it by hanging the vial from the spring and reading the position of the bottom of the spring. Record this reading in Table I, left-hand side, as "No load position." Then detach the vial and take it back to your teacher to obtain a measured sample of about 10 cc of water, added directly to the sugar in the vial. Again, the exact amount of water is unimportant but must be read accurately to 0.01 cc. The amount should be between 9.5 and 10 cc. Record the volume received in Table I.

Now cap the vial again and gently swirl the water and sugar around until the sugar is completely dissolved. Uncap the vial and look at the contents from time to time to find out whether the sugar is completely dissolved. It must be completely dissolved before proceeding with the experiment. The larger amounts of sugar dissolve only slowly (perhaps 10 minutes or more of constant swirling); this is why the larger amounts were given out first. Do not shake the vial, because some of the material will then become trapped between cap and vial. Meantime, attach the second vial, capped, to the spring and read again the spring position. Record this reading at "No load position" in the right-hand column of Table I.

When the sugar has completely dissolved, take both vials to your teacher, and pipette out of the vial that contains the solution, about 10 cc of solution. Read the volume to the nearest 0.01 cc and then drain the contents of the pipette into the other empty vial. Record the volume of solution in the right-hand column of Table I. Cap the vial containing the solution you just pipetted into it, and leave the other cap and vial with your teacher. Take the vial with pipetted solution back to your weighing machine, hang the vial on the spring, read the spring position, and record this position in the right-hand column. You are now finished obtaining all the data you need. Clean and put away your apparatus and get ready to do some calculating.

Calculating From your Data: The calculations in this experiment are more involved than you are accustomed to, and perhaps you should be guided through them. The following paragraphs will explain how you can calculate the concentration and the density of the solution you made. Please notice that this is an explanation. It would be possible to tell you what to do, you could dutifully go ahead and do it, get it entirely right --- and have learned nothing. But that's not the way we do things here. Notice that each step follows logically one after another. You make the calculations in the way explained below, not because somebody says this is the way to do it; you do it this way because logic says this is the way to do it. Keep your mind open and try to understand why each step is taken. It would be a good idea to read the whole thing once before you start calculating, just so you can see the flow of the whole idea. Ready?

Remember that you want to calculate the concentration and density of the solution. Keep this in mind, because you have to know where you are going before the directions for getting there make any sense.

We'll start with the concentration of your solution. First, what do you mean by concentration? If you don't know exactly what it means, don't you think you should go back and look it up: how can you expect to understand how to calculate concentration when you don't even know what it is?! Refer back to the end of the introductory paragraph of this experiment. Now you know that the concentration of your solution means the number of grams of sugar per cc of solution. How can you get this? You made up the solution by taking some sugar and dissolving it in water. You then can get the concentration of the solution by dividing the weight of the sugar you used by the volume of solution produced. You can easily get the weight of sugar you used: You have the no load spring position and the spring position with the sugar added; from these you can get the spring extension; and then, knowing the spring constant, you can get the weight of sugar. Do this, using the left-hand part of Table I to record your data. Now you have the weight of sugar in your solution, but ---

You also need the volume of the same portion of solution that contains that weight. You cannot get the concentration by dividing the weight of sugar contained in one portion of solution by the volume of some other portion. The weight of sugar contained in all the solution you prepared -- not just part of it. Moreover, the volume of the solution is not merely the volume of water used to make it, because the sugar, even when it's dissolved, takes up some room, too. One way to get the volume of the total solution would be to transfer it totally to a pipette, but this would be difficult to do without leaving behind some droplets -- or at least some wetness -- in the vial. There is another -- indirect -- way to find this total volume.

You can find the total weight of the solution, can't you? The total weight is simply the sum of the weights of the sugar and of the water that you combined to make the solution. You just calculated the weight of sugar in the paragraph above. OK,. add the weight of water to it. But wait a moment; you didn't weigh the water! But you did measure its volume; can you get the weight if you know the volume? Of course; all you need is the density of water, which is the same today as it was when you did Experiment 18. Look up the density of water from Experiment 18 and then calculate the weight of water added. (You could have weighed the water -- or obtained the total weight of water and sugar together -- directly by weighing them with your spring.) Enter the weight of water in the last line of the left-hand portion of Table I. Then add the weight of water to the weight of sugar and enter the sum as "weight of solution" on the first line of the centered bottom portion of the table.

But to calculate the concentration of the solution, you need its volume, not its weight. How can you find the volume from the weight? For this calculation you need the density of the solution. Do you know the density of the solution? No, not yet; but you have the necessary data from which you can calculate the density. These data are in the right-hand portion of Table I.

To calculate the density of the solution, you need the weight and the volume of some sample of the solution. To find the density, is it necessary to work with the whole solution? No; because you remember that the density of a fixed material is always the same regardless of the size of the sample. Hence we can find the density of the total solution by working with any size portion of it we please. This is what saves us, for then it is not necessary to be sure to transfer the whole sample to the pipette without leaving any wetness behind. We can go ahead and make up the solution out of carefully measured components, and then find the density of a convenient portion of it without worrying about complete transfer, and still be assured that we have the density of the whole solution. Of course, it is best to use as large a portion of the solution as you can conveniently get to make the measurements for the same reason as was discussed on page 98 .

From the two spring positions recorded in the right-hand part of Table I, you can calculate the weight of your sample of solution. Do this, and enter both "extension" and "weight of sample" in the table. Knowing the volume, you can calculate the density. Do so. This will be the last entry on the right-hand side of the table.

Now you know the density of the solution. Since you have already found the weight of solution (first line of the bottom portion of the table), the density and weight together will allow you to calculate the volume. Do so, and enter the result in the second last line of the table.

To get the concentration of the solution, you need the weight of the sugar contained in a known volume of solution. But you know the total weight of sugar used and you now also know the total volume of solution you made -- even though you never measured it directly. Calculate the concentration and enter it in the last line of the table.

Now you are finished!

That wasn't so hard, was it? It was long, of course, but it wasn't hard. Have you ever looked at a piece of chain and pictured clearly how it holds together? It isn't very hard to see how a chain works -- how each link encircles its neighboring links in such a way as to produce a whole train of hold-together links. Would you say that a long chain of 100 links is more complicated than a short chain of five links? Of course not! The long chain is merely longer -- not more complicated.

The long chain of reasoning that you just went through is long, of course; but it is not complicated. You can easily understand every link in the chain. Don't be worried about the possibility that you might need help in putting the links together. Everybody needs help at first, and you are only beginning your study of physical science. Learn the little things first so that you understand them. The big things will later follow easily.

The Graph: You have worked out one pair of values, density and concentration, for one sugar solution. Copy your calculated values of these two quantities into Table II, the second line. Others in your class will have worked out similar data for other solutions. Your teacher will arrange things so that everybody's results will be available to everybody else. Copy everybody else's results into Table II, and then you will have a whole set of densities and concentrations. Notice that the first line is already entered for you; do you understand its meaning? Pure water is really a sugar solution having zero concentration, isn't it?

Now plot the data of Table II on the graph at the bottom of the page. Notice that the origin does not appear on this graph. The reason for this is simply that putting in the origin would make most of the graph merely blank space.

Experiment 20
Data Sheet #1

Table I

Calculation of Concentration and Density for One Sample of Sugar Solution

Spring Constant __________ **g/cm**

Total Solution	Sample of Solution
No load Position ____________ cm	No load Position ____________ cm
Pos'n with sugar ____________ cm	Position with Solution __________ cm
Extension ________________ cm	Extension ________________ cm
Wt. of sugar ______________ g	Wt. of Sample ______________ g
Volume of water ____________ cc	Vol. of Sample ______________ cc
Wt. of water ______________ g	Density of Solution __________ g/cc

Wt. of Solution __________ **g**

Volume of solution __________ **cc**

Concentration ________ **g/cc**

Experiment 20
Data & Graph Sheet #2

Table II

Concentrations and Densities for Sugar Solutions

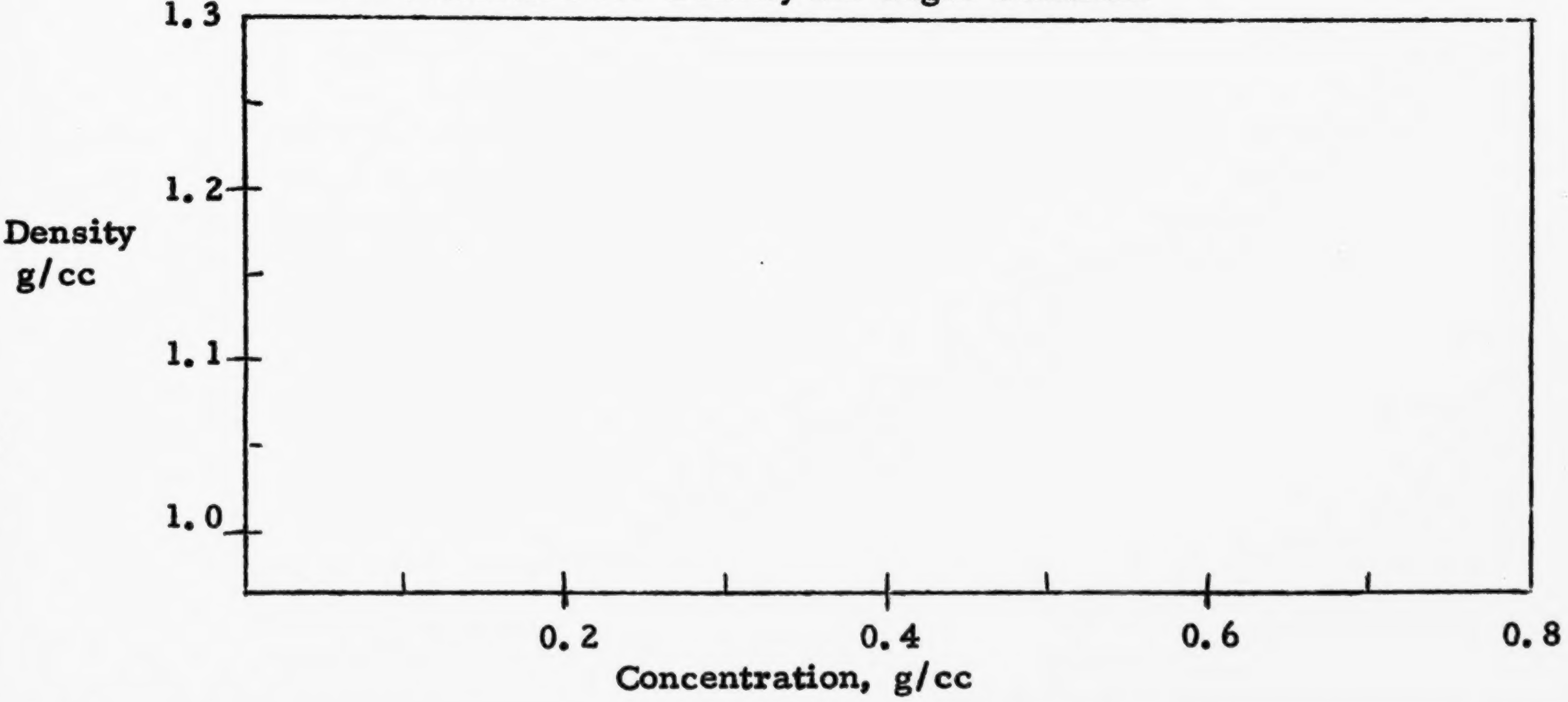
Conc g/cc	Dens g/cc
0.000	0.997

Conc g/cc	Dens g/cc

Conc g/cc	Dens g/cc

Graph

Concentration vs. Density for Sugar Solutions



Experiment 21

Motion under Constant Velocity (Part 1)

The purpose of this experiment is to look into the interconnection among the quantities of time, position, distance, and velocity for a body moving along one straight line. To do so, we need to get our hands on something that can be trusted to move with constant velocity and slowly enough to make satisfactory measurements. One way to do this is to use the fact that a body falling under gravity through a fluid will eventually reach a constant velocity if it falls far enough. A ball not too heavy or large falling through a heavy oil reaches this velocity after falling only a centimeter or less. That is the way constant velocity will be assured in this experiment.

Procedure: The entire class will perform this experiment together. Seven persons are needed at one time to perform the experiment. Your teacher will assign duties to a team-of-seven who will carry out the experiment; another team to do it again; and so forth.

The preparations described in this paragraph and the next will be completed ahead of time. First, a strip of paper about half an inch wide and seven feet long should be obtained: a strip cut from an adding-machine roll would do nicely; or an ordinary sheet of paper cut into strips securely scotch-taped together will do as well. This should be made into a measuring tape as follows. Lay the strip down running from left to right and about one fourth of the way from the right-hand end, make a short mark like those on a ruler. Label this mark zero. Then lay off to the left of this mark a series of other ruler-marks ten cm apart. Label them in order from zero to the left, 10, 20, 3, etc. up to 150. Label them in plack pencil, the black signifying plus. Then lay off a similar series to the right, labeled from zero to the right in order, 10 up to 50. These should be labeled in red pencil, the red signifying minus. You now have a eentimeter measuring tape marked + and - from zero, the zero not being in the middle. On this tape, plus is on the left and minus on the right.

Second, a long glass tube about 2 cm in diameter and 120 cm long is securely stoppered at one end, and securely held vertical (stoppered end down) by tying it with thread to a dowel post set in a breadboard. This is filled within a few centimeters of the top with a heavy mineral oil like, say, Nujol. This should be allowed to stand overnight in order to come to a uniform temperature.

You are ready to start. Arrange the tape-measure vertically along the outside of the tube so the -40 cm mark (red) is about 5 centimeters below the liquid level in the tube, scotch tape the upper end securely, stretch the measuring tape vertically downward along the tube, and scotchtape it again near the bottom. Never mind the excess measuring tape at the bottom; it will be too long and some "unused" tape will just lie there unused. Leave it there.

Now we need a team of seven. Their jobs are:

2 Timers 2 Recorders 2 Readers Dropper

The dropper has a supply of plastic beads. He stands holding a bead in his fingers over the top of the tube.

The timers stand where they can read a clock with a second hand and keep their eye on the clock.

The recorders stand near the timers (one recorder paired with each timer) with pencil and paper. The papers have previously prepared tables of five columns and about 8 lines. One recorder has the even multiples of 10 entered in the first column: -40, -20, 0, +20, and so on down to the last mark on the tape. The other recorder has the odd multiples: -30, -10, +10, +30, and so on. The recorders are designated odd and even. The tables may be on scratch paper. The recorders hold pencil in hand and keep their eyes on the paper, listening each to his own timer.

The readers stand one on each side of the oil-filled tube, in a position where they can clearly see both the paper measuring tape and a ball falling down through the liquid at the same time. One reader is teamed with one timer-recorder pair and the other with the other. The two readers should have readily distinguishable voices so the timer and recorder can tell without looking which reader is speaking. It might help, for instance if one reader is a girl and the other a boy. It is the readers' job to watch the ball as it falls through the oil. They must keep their eyes on the ball. One reader is designated odd and the other even, and are so teamed with the corresponding recorders.

Here is the performance; everybody ready? One of the timers watches for the approach of the second hand to the 12 (that is, zero) on the clock. He announces 10 seconds to go, then counts down 5, 4, 3, 2, 1, GO, calling GO when the second hand is at 12. At GO, the dropper drops a ball into the oil, releasing it just under the surface. (Of course he gets his fingers oily, but he has nothing else to do anyway.) The readers watch the ball descend.

When the ball comes exactly opposite the -40 mark on the tape, the even reader calls out in a staccoto voice "Four". The even timer, eyes always on the clock, reads the position of the second hand on the clock at the moment he hears "Four". The hand will be moving, of course, and he must very quickly make up his mind where the hand was at the moment he hears the signal. He estimates the time to the nearest 0.1 second. Without taking his eyes off the clock, he announces the reading quietly to his recorder who records the reading in the second column of his table, on the line where "-40" appears in the first column.

Meantime the odd reader, eyes always on the ball, watches for the moment when the ball comes exactly opposite the -30 mark. At this moment he calls "Three" in a staccato voice, and the odd timer, eyes always on the clock, reads the position of the second hand at the moment he hears the signal. He makes this time reading, like all the rest, to the nearest 0.1 second, and announces it immediately and quietly to his recorder, who records the reading in the second column of his table on the line where the first column says "-30".

Meantime the even team, it is to be hoped, has recovered from its -40 task. The even reader continues to watch the ball. When it comes exactly to -20, he calls "Two" and the even timer reads the clock and announces the time to his recorder, who records the time in the second column opposite "-20". By this time the odd team will have recovered from its -30 activity and makes a reading for -10. Then the even team at zero, odd at +10, and so on until the ball falls to the bottom. The only persons in the entire class permitted to talk are the readers and timers. This is very important.

The whole team should make several practice runs so they can work together as a team. They then make three runs for the record. The recorders will then have three readings of the time for each multiple-of-ten position of the ball. They should average each set of three readings to get a "best" time for each position. The two lists of averages are then blended into one sequence of averaged times when the ball was at -40, -30, -20, and so on up to +80 or whatever the bottom reading turns out to be. These data are entered in the first two columns of Table I on the data sheet.

The team now retires to a well-deserved rest. Their places are taken by another team with assignments exactly as before. First, the measuring tape is detached and moved upward so the zero mark (instead of -40) is about 5 cm below the liquid level. Secure it in place. The recorders make their data columns with the first column reading 0, 20, 40, etc., to the bottom reading of the tape for the even recorder; and 10, 30, 50, etc. for the odd. Repeat the entire performance as before and enter the averaged and blended data in the third and fourth columns of the data sheet. Second team is now finished.

Another set of runs should now be made with a new team. Detach the tape and reattach it with the +40 (black) mark about 5 cm below the liquid level. Recorders prepare the first column of the data sheet with +40 (even) or +50 (odd) as the first entry. The final averaged and blended data are entered in the last two columns of the data sheet.

Finally, make plots (time horizontally and position vertically) of all three sets of data on the same graph on the second graph sheet. The positions (in cm) are already marked on the vertical axis, but you will have to supply your own time scale. Plot positive (black) distances above the zero and negative (red) below. (This is opposite to the physical direction of fall.)

Leave the "calculated velocities" blank for now.

Experiment 21
Data Sheet #1

Table I

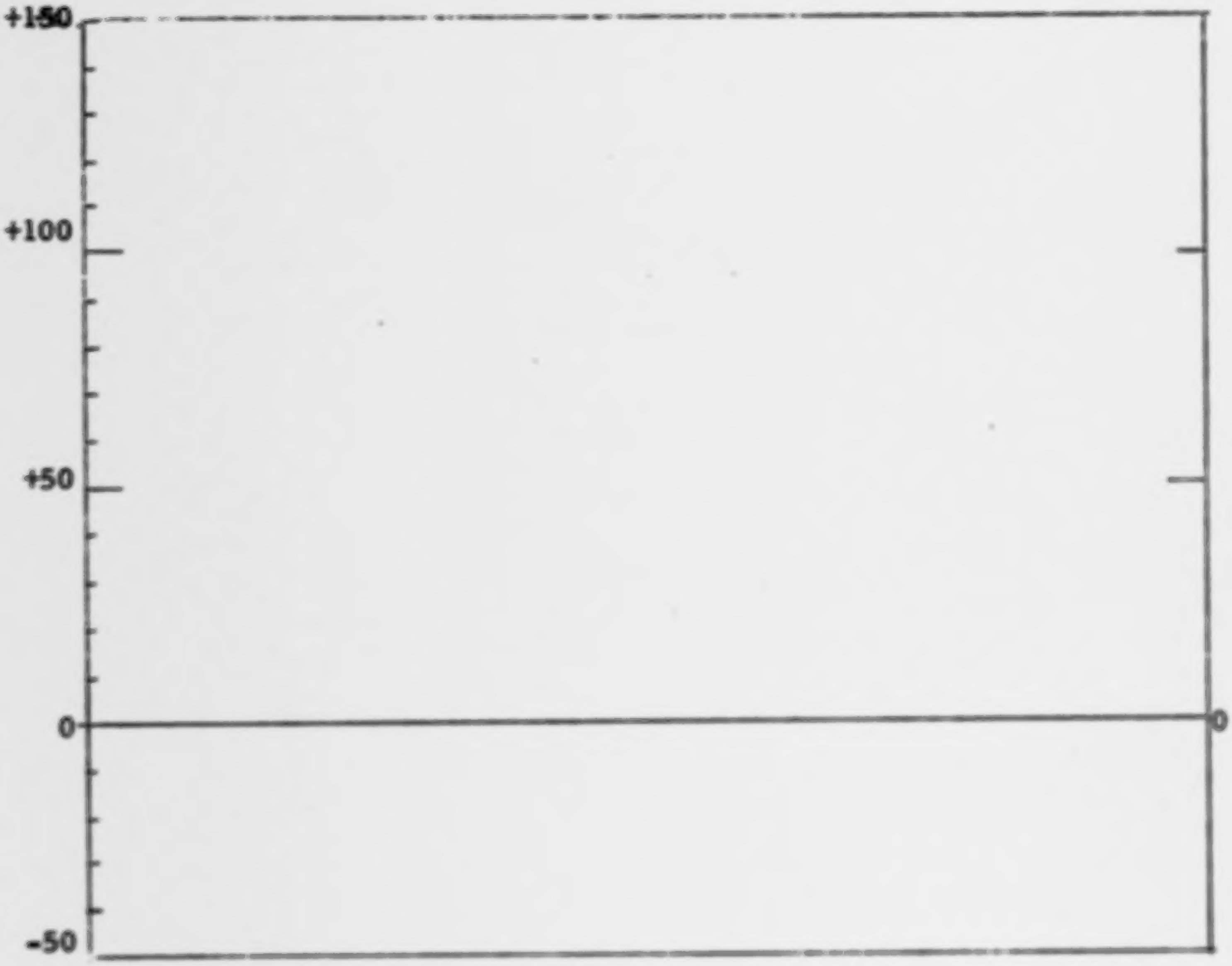
Position vs. Time in Uniform Motion

First Run		Second Run		Third Run	
Position cm	Time sec	Position cm	Time sec	Position cm	Time sec
-40.0		0.0		40.0	
-30.0		10.0c		50.0	

	1st Run	2nd Run	3rd Run
Calculated velocities	cm/sec	cm/sec	cm/sec

Experiment 21
Graph Sheet #2

Position vs. Time in Uniform Motion



Time, seconds

Experiment 22

Motion under Constant Velocity (Part II)

You will remember that Experiment 22 was carried out by dropping balls of the same size and same material through the same column of oil. Your intuition would tell you that the balls would all fall with the same velocity. This in fact is true -- they do all fall with the same velocity.

We now wish to compare the functional relationship between position and time for bodies falling with different velocities. You would probably guess that if you dropped a bigger ball through the liquid, it would fall with a different velocity. This also is true, and this is how we will obtain different moving bodies, each with a different, but constant, velocity.

Procedure: The setup and procedure, including team assignments, are identical with those of Experiment 21, except that the droppers will use balls of different sizes from those in the preceding experiment. Adjust the measuring tape so that the zero-mark is about 5 centimeters below the liquid level and leave it there for the entire experiment.

The first team repeats its performance in Experiment 21 with the tape in the position noted in the preceding paragraph, except that the dropper uses balls a little bit larger than those in Experiment 21. The team again makes three runs for the record. The data are averaged and blended as before, and the readings recorded in Table II. The first column of Table II lists the tape-positions beginning with 0.0 cm and increasing by tens to the bottom of the tube. Since the tape will not be moved during this experiment, this column will serve for all the trials. The averaged times-of-passing for the 10-cm marks are to be recorded in the third column; leave the second column blank for now.

The second team repeats the performance of the first team, leaving the tape unchanged, but using balls a bit larger still. The data are treated in the same way and the times-of-passing are recorded in the fourth column of Table II.

If time permits, a third team should make still another series of runs using a set of balls still larger. These data, if taken, should be recorded in the fifth column.

Now fill in the second column of Table II. These data are the times-of-passing the 10-cm marks for the balls used in Experiment 21. Simply copy the data fromthe fourth column of Table I (Experiment 21), since there is no point in doing that experiment over again.

Then make plots (time horizontally and position vertically) of the data in Table II. All distances are positive this time. Plot the points all on one graph on the second work sheet. The data copied from Experiment 21 are also to be plotted on this graph, and of course will merely be a copy of what you plotted before. Use a ruler to draw the three (or four, if the third team performed) straight-line curves. As usual draw the lines so you leave about as many points on one side of the line as on the other.

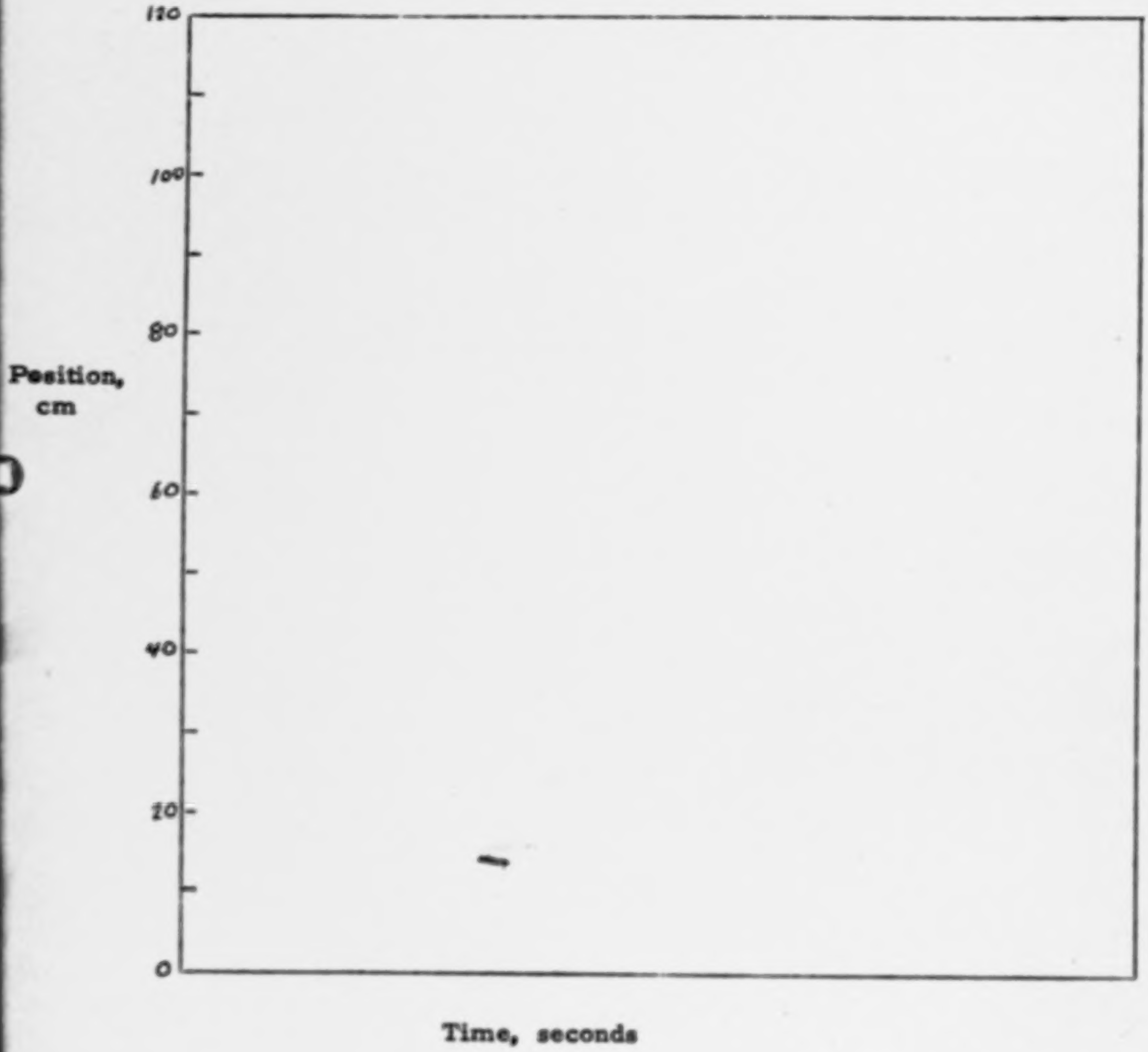
Experiment 22
Data Sheet #1

Position vs. Time in Uniform Motion
Table II

	Slow	Medium	Fast	Faster
Position, cm	Time sec	Time sec	Time sec	Time sec
0.0				
10.0				

	1st Run	2nd Run	3rd Run	4th Run
Calculated velocities	cm/sec	cm/sec	cm/sec	cm/sec

Experiment 22
Graph Sheet #2



Position vs. Time in Uniform Motion

Experiment 23

Motion under Constant Acceleration

The purpose of this Experiment is to investigate the motion of a body in a simple case of accelerated motion. We will use a ball rolling down an inclined ramp as an example of accelerated motion.

There is a trouble with this experiment of which you should be forewarned. If the ball rolls too slowly, then friction takes charge and the motion quickly becomes constant-velocity motion like the ball slowly falling through oil. If the ball rolls rapidly, its motion is truly uniformly acclerated, but then the action is so fast that it is difficult to measure. In truth, the slowest movement that can safely be used to avoid the effect of friction is still too fast to get excellent results. This experiment therefore is an example of something one often meets in experimental physical science: you make measurements which you recognize as being of low precision, yet treat the data in such a way as to average out the errors as well as you can. This experiment, it might be added, can be carried out with high precision, but the equipment required to do it would be prohibitively complicated. Perhaps after you have completed the experiment you will be able to suggest more complicated measures that might be taken to secure better precision.

Procedure: A 12-foot long 2 x 4 wooden plank will be provided. It has a 1/2-inch wide and 1/2-inch deep groove cut along its entire length in one of the narrow ("two inch") faces. The board should be as free of warpage as possible, and the dadoed groove should have clean edges fairly free of splintered-out sections. The edges of the groove will provide the track for a steel bearing-ball to run along.

Lay the board out on the floor, grooved edge up. Prop up one end of the board to make an inclined ramp for the ball to roll down. A stack of books about 25 cm high under one end will make the ramp steep enough to give uniformly accelerated motion, yet not so steep as to make the speed of the ball impossibly fast for measurement.

You will also need an electric clock having a sweep second hand, or a stopwatch. If you use an electric clock, look at the scale marks on the dial. On many clocks, made to be viewed from a distance, the minute (or second) marks are thick lines as wide as the space between them. Imagine how difficult it would be to read a ruler on which the centimeter "lines" were half a centimeter wide! If your clock is made like this, it may be necessary to modify it by pasting a make-shift scale made of a narrow arc of paper over at least a part of the clock scale, and making thin pencil marks at the centers of the dial's own thick ones. If this is necessary, your teacher will have done it ahead of time.

A steel bearing ball about 2 cm in diameter will also be provided. Make a pencil mark across the grooved face of the board about 2 cm from the upper end. and label it "O", then use a meter stick and a pencil to make further marks across the grooved face of the board, every 40 cm from the mark at the upper end. Label these marks 40, 80, etc. down to 360 cm. The "O" mark is the origin, and we will again take downward as the positive direction.

This is a team performance, and it is probably better to do the entire experiment with one team given 10 minutes or so to rehearse than to use several teams each of which will have to spend an equal time in practice. The team consists of:

One starter, two timers, two markers, and one retriever.

The starter stations himself at the upper end of the board. He places the ball in the groove and holds it there lightly with his finger, with the leading edge of the ball opposite the starting mark onthe board. When No. 1 timer calls "Go", his sole job is to release the ball. NO pushing; simply lift the finger and let the ball start slowly by itself.

The two timers stand where they can clearly see the clock. They must be able to get within normal reading distance of the clock; say, about one foot. No. 1 timer stands immediately in front of the clockface with No. 2 close by his side. They both keep their eyes on the clock. No. 1 timer watches the second-hand, and when it approaches a part of the dial-scale he can easily read, he warns the starter: "Get ready... GO", calling "go" at the instant the second-hand passes a convenient scale-makring. He stays there with eyes always on the moving secondhand, waiting for No. 1 marker to call "Mark." When he hears the signal, "Mark", he makes a reading of the second hand, judging the time to the nearest 0.1 second. He immediately moves aside. No. 2 timer, who has never taken his eyes off the second hand, then immediately moves in front of the clock face and waits for NO. 2 marker to call "Mark". At this signal, he too reads the position of the second-hand to the nearest 0.1 second.

It requires quick eyes and quick decision to judge the position of the moving second hand this way. It is not otherwise difficult, but the timers must be able to make up their minds quickly. After both readings have been made, the two timers announce their readings to the class. Half the class will act as recorders for No. 1 timer and the other half for No. 2. They note on scratch paper the times announced by the timers.

The two markers station themselves along the side of the ramp, No. 1 marker at the 40-cm mark and No. 2 marker at the 240- cm mark. Markers keep their eyes mostly on these marks. When they hear the word "Go", they watch their marks carefully, waiting for the arrival of the ball. At the instant when the ball hits the 40-cm mark, marker No. 1 calls "Mark", the signal that timer No. 1 above was waiting for. He does not disturb the ball in any way -- merely announces its arrival at the 40-cm mark. Meantime, marker No. 2 awaits the arrival of the ball at the 240-cm mark, announcing its arrival there by calling "Mark" also. This is the signal that timer No. 2 above was waiting for. The ball continues its way down the ramp where it is caught by the retriever who carries it back to the starter.

The experiment is repeated at least five times, so that five readings of the times of passage to the 40-cm and 240-cm marks are obtained. The members of the class doing the recording should now average the five time readings they recorded and some member of each recording group should go to the blackboard and enter the average time for passage at 40 cm and at 240 cm in a table like Table I on the data sheet. The averages should be taken to two decimal places, though the last figure will be very unreliable.

A peculiarity of repetitive experimental measurement will arise with the timers. It is called "experimental prejudice" and is extremely difficult to avoid, even by skilled veteran scientists. It is simply this. If a timer reads the first time as "1.7 seconds", he will find it very difficult to forget the fact. On repeating the measurement on the second roll, the call of "mark" may occur when the clock hand actually is at 1.8 or 1.8 seconds. But the timer, remembering that he judged it as 1.7 the first time will find it extremely difficult to resist calling it 1.7 again, simply because he read it that way the first time and feels that the second time ought to give the same reading as the first. Try to make each reading uninfluenced by preceding ones. You will find it hard to do.

The reverse of the effect mentioned in the preceding paragraph is just as likely to occur and just as bad. A timer may read the first time as 1.7 seconds. The second reading may also be 1.7 seconds. But the timer, remembering that his first reading was 1.7, may resist calling the second one 1.7 also because he thinks he is being influenced by the first when he really is not. The best thing is to keep in mind that the clock has no memory, and the timer who reads it should have no memory either.

After five readings of passage at 40 cm and 240 cm are taken, the markers should move downhill to the 80-cm and 280-cm marks and the team repeats the whole performance, taking 5 readings of each again. Then repeat with the markers at 120-cm and 320-cm (five readings); then 160-cm and 360-cm (five readings); then marker No. 2 and timer No. 2 retire, and marker No. 1 calls "mark" for passage of the 200-cm mark. The data are displayed on the blackboard and the entire class then copies the complete table as the first two columns of Table I on the first data sheet.

You should now make a graph in the upper space of the second work sheet, plotting position (the dependent variable) vertically and time (the independent variable) horizontally. Leave the other graph frame and data columns blank for now.

Experiment 23
Data Sheet #1

Table I

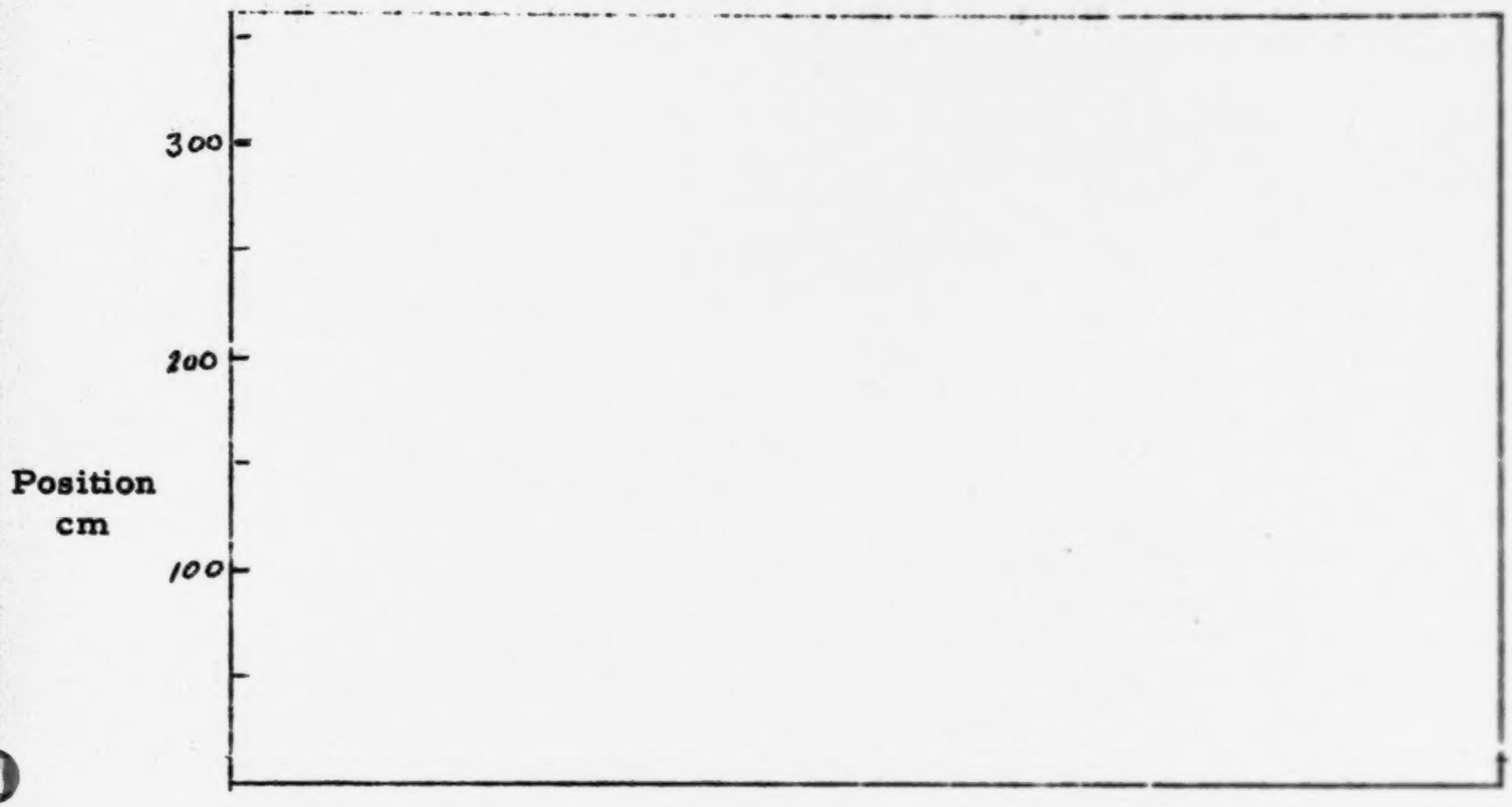
Motion under Constant Acceleration

Pos'n	Time, t, seconds	
p, cm	meas'd	Smooth
0	0	0
40		
80		
120		
160		
200		
240		
280		
320		
360		

p	t	V	Midway		Ratio
cm	sec	cm/sec	Pos'n cm	Time sec	v/t cm/sec^2
40			20		
40					
40					
40					
40					
40					
40					
40					
40					

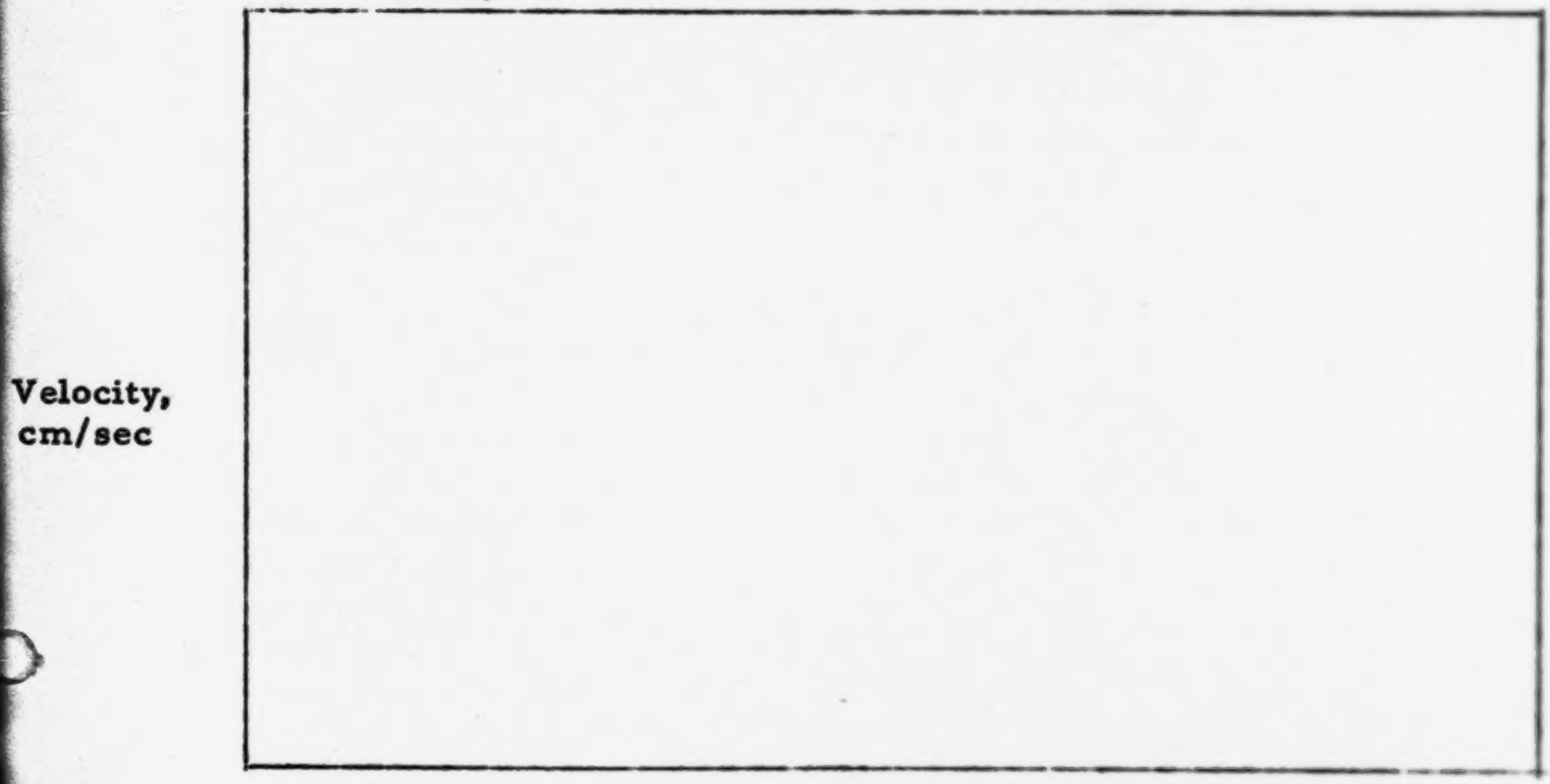
Experiment 23
Graph Sheet #2

Position vs. Time for Constant Acceleration



Time, Seconds

Velocity vs. Time for Constant Acceleration



Time, Seconds

Experiment 24

Tangent to a Curve

This experiment will give you a little experience with the notion of "tangent to a curve at a point." You will use the curve of p vs t obtained in Experiment 23. You will obtain the tangents in two different ways, and at the same time find a simple relationship between p and t that you may not previously have suspected.

Recall that you obtained, in Experiment 23, the times, t, that it took a ball to pass various positions, p, as it rolled downhill. You have these data in Table I of that experiment. Refer back to this table and look at the first and third columns. The first column gives the series of positions at which you measured the times of passage, and the third column gives the smoothed values of the times.

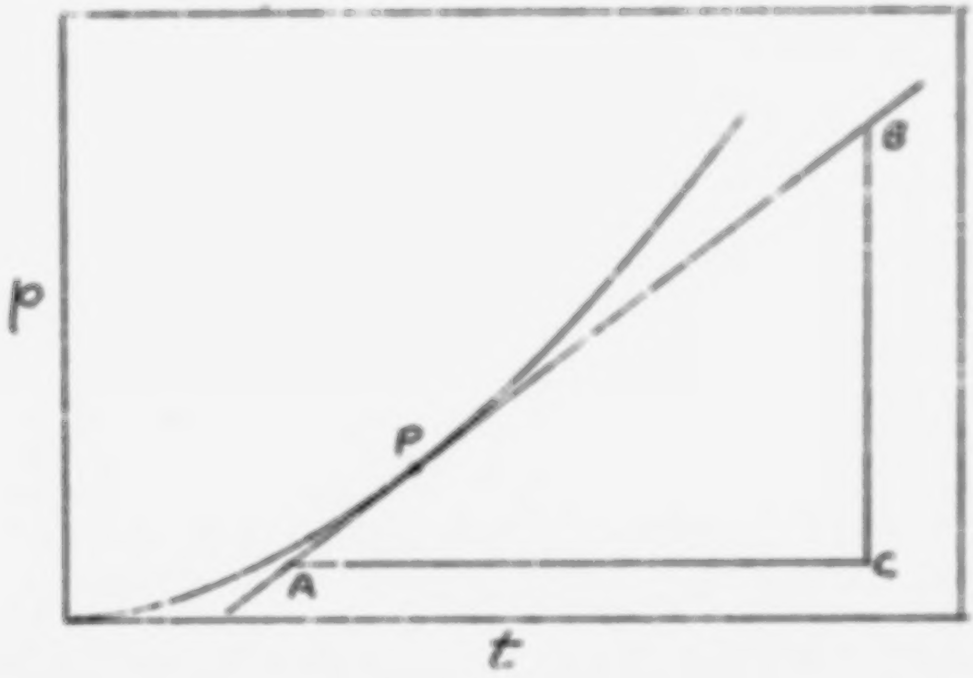
Procedure: Make another graph of p vs t on the first work sheet for this experiment. Do this by plotting horizontally the times (third column of Table I in Experiment 23) and vertically the positions (first column) for the ball rolling downhill. Your graph will of course merely be a duplicate of your first graph in Experiment 23, but somewhat larger. Draw in the curve connecting the plotted points. Do this with the best care you can take. The curve must be smooth (no wiggles) and cleanly drawn. Have your teacher approve your drawing before you go ahead. Now erase the plotted points and fill in again the gaps in the curve created by erasing.

Make a small penciled dot on the curve at a value of p equal to 140 cm. We will refer to this point as P. You are going to draw a tangent to the curve at this point. Before doing so, however, you should play around a little. Take your ruler and lay it across the curve like a secant, cutting the curve at P and some other point farther to the right. We will call this other point Q, but do not bother to mark it or label it. Notice the slope of the secant ruler.

Now use your left hand to keep the ruler with its edge passing through P, and rotate the ruler slowly clockwise around P and notice how the other point, Q, moves slowly to the left and downward along the curve. Notice at the same time how the slope of the secant-ruler constantly changes as Q moves toward the fixed point, P. Lay the secant-ruler across the curve so Q lies to the left of P and repeat, this time rotating the ruler counterclockwise around P. Again notice how the slope changes as Q moves closer and closer to P.

You now see that the second intersection, Q, may lies either to the right or to the left of P, depending on how much the ruler is tilted. You can also see that the ruler may be moved so that Q comes as close to P as you please. Can you arrange the secant-ruler so that it passes through P only, through no other point on the curve, yet does not cross over the curve? This, of course, is the position of the secant-ruler when Q has been moved so close to P that Q lies right on top of P. Think of the curve as the rasied curved curbing around a street corner and the ruler as a long board; you are in the street and moving the long board horizontally up to the curb so that it just touches.

When you think you have the right idea, adjust the ruler so that it is tangent to the curve at P and actually draw the tangent with a pencil using the ruler as a straight-edge guide. Be sure the pencil line goes through the point P. Waggle the ruler back and forth a little before you draw the line to be sure you have it just right. Draw the tangent long enough so that at least 15 cm of it lies inside the frame-lines of the graph. Your drawing will look something like this:



Choose any two definite points, say A and B, that lie on the tangent line and are at least 15 cm apart. Then draw the horizontal line AC and the vertical line BC. Be sure AC and BC are truly parallel to the axes of the graph. You will now have to measure $\overline{AC}$ and $\overline{BC}$. But there is a catch involved in measuring them; do you see what it is?

You must remember that $\overline{AC}$ represent Δt. It therefore represent some number of seconds, and is not to be measured in length units like cm. Its "length" must be measured in seconds, using the same scale that was used to plot the graph. You therefore have to measure the length of $\overline{AC}$ using the t-scale at the bottom of the graph as a ruler. One way to do this is to take a sheet of paper and lay its edge along the line AC, and mark the edge of the paper with one tick exactly at A and another exactly at C. Then move the paper so that the edge lies along the t-scale (bottom frame line) of the graph, with the left hand tick at the origin. Read the position of the right-hand tick against the t-scale just as you would any other ruler. This reading is the value of Δt, in seconds, which should be recorded in the fourth column of Table I of the present experiment. Measure $\overline{BC}$ in the same way, but use the vertical p-scale (left frame line) as the ruler this time. This is Δp and should be recorded in the fifth column of the table.

The point P above was marked at $p = 140$ cm. You should now repeat the whole performance for other points P of the curve. Do at least three others; measure Δp and Δt for tangents at $p = 60$, 220, and 300 cm first, and then others if you have time. If you do others, choose p's listed in the first column of the table.

Columns four and five of this table now give you Δp and Δt for tangents to your p vs t curve at several different points on the curve. Calculate $\Delta p/\Delta t$ and record the ratios in the sixth column. Also, complete the first three columes of the table. The first column, already filled in, gives a series of positions of a ball rolling downhill. The second column gives the times at which the ball passed these positions. You are to obtain these times by reading them from your graph -- values of t at $p = 20, 60, 100, \ldots,$ 340 cm. The third column is the average velocity, $\bar{v}$, that you previously calculated for a small interval surrounding p - 20, 60, etc. cm. Copy these values from Table I of Experiment 23.

You should now have a classroom discussion before proceeding with the rest of this experiment.

Experiment 24
Work sheet #1

Tangents to Curve p vs. t

Experiment 24
Data Sheet #2

Table I

Velocities under Uniformly Accelerated Motion

From Expt. 23			From tangent slopes			From derivative		
Pos'n, p cm	Time, t sec	$\overline{V}$ cm/sec	Δp cm	Δt sec	v cm/sec	t^2 sec^2	p/t^2 cm/sec^2	V cm/sec
0	0	0	0	any thing	0	0		
20								
60								
100								
140								
180								
220								
260								
300								

k = average p/t^2	
acceleration	